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JOURNAL OF THE GEOTECHNICAL ENGINEERING DIVISION

Extreme-Value Problems of Limiting Equilibrium

By Michael Garber[1] and Rafael Baker[2]

Introduction

Many of the problems encountered in Soil Mechanics are of the extreme-value type. Such problems are, e.g., the stability of slopes, the bearing capacity of foundations, the limiting forces (active and passive) acting on retaining structures, etc. These problems and a number of similar ones are shown in Fig. 1. In each of these problems it is required to find the extreme (minimal or maximal) value X_{ex} of some parameter X, while all other parameters defining the problem are assumed to be known. According to the character of the problem, X may be one of the parameters F, P, x_p, y_p, β, or M, in which F = the factor of safety with respect to strength; P = an external load; x_p, y_p = the coordinates of the point of P application; β = the direction of P; and M = an external moment.

All the problems described in Fig. 1 may be solved within the framework of the limiting equilibrium (LE) approach. This approach, which considers a "test-body" bounded by soil surface $\bar{y}(x)$ and slip surface $y(x)$ (see Fig. 2) is based on the following three concepts: (1) Satisfaction of failure criteria $\tau = f(\sigma)$ along the slip surface, in which $\tau(x)$ and $\sigma(x)$ are distributions of the shear and normal stresses along $y(x)$, respectively; (2) satisfaction of all equilibrium equations for the "test-body"; (3) extremization of factor X with respect to two unknown functions $y(x)$ and $\sigma(x)$ so that, in fact, X is considered as a functional of these functions. The extreme value, X_{ex}, is defined as

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$$X_{ex} = \underset{\substack{y(x)\\ \sigma(x)}}{\mathrm{Extr}}\, X[y(x), \sigma(x)]$$

The existing methods differ from one another in the assumptions about the character of previously mentioned functions $y(x)$ and $\sigma(x)$. The straight line, circular arc, and log-spiral are the most widely used assumptions with respect to the character of $y(x)$. The form of $\sigma(x)$ is either assumed directly or introduced indirectly by assumptions regarding the nature of the interaction between sections of the sliding mass. However, if the aforementioned assumptions regarding $y(x)$ may be justified by some experimental observations, then the popular assumptions regarding $\sigma(x)$ are considerably more arbitrary. Thus, we must recognize that the majority of existing methods are poorly argued. As a result, we cannot

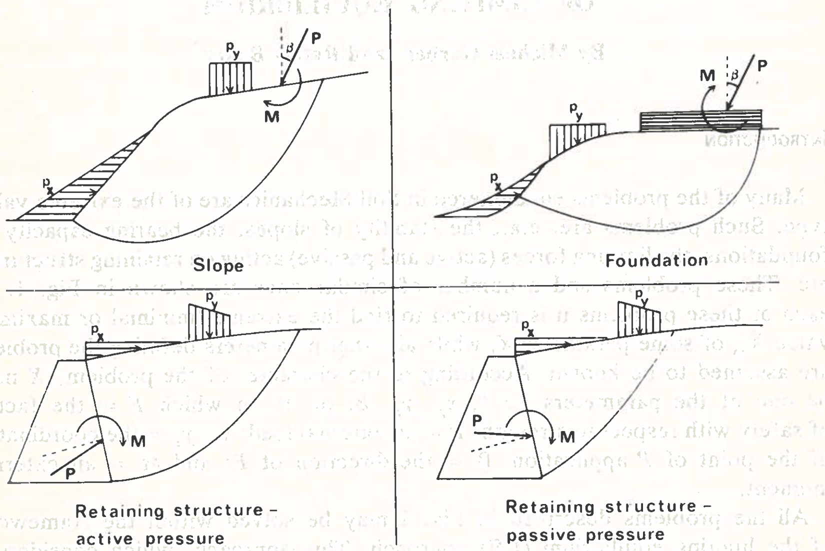

FIG. 1.—Different Extreme Value Problems: $X = F$—Slope Stability; $X = P$—Limiting Load; $X = x_p$—Limiting Location; $X = \beta$—Limiting Direction; $X = M$—Limiting Moment

apply them with sufficient confidence. Finally, we cannot conclude in any specific case which one of these methods is the most justified.

It appears that the proper way to get out of this situation is to base the analysis on the concepts of LE only. Several attempts to treat the stability problem in such a way have been made (2,5,6). All these attempts deal with the case of homogeneous and isotropic soil. The first attempt to analyze the stability problems (slope and foundation) in terms of two unspecified functions $y(x)$ and $\sigma(x)$ was made by Kopacsy (6) in 1955. However, a reappraisal of Kopacsy's analysis by Baker and Garber (1) shows that it contains a number of serious errors and misconceptions.

The analysis of the slope stability problem in terms of two unspecified functions $y(x)$ and $\sigma(x)$, using the usually accepted factor of safety with respect to strength,

was performed first by Baker and Garber (2). The analysis and the solution of the bearing-capacity problem, in which $y(x)$ and $\sigma(x)$ are unspecified and the factor to be minimized is the foundation load, were presented first by Garber and Baker (5). Both works deal with the case of homogeneous and isotropic soil. The analysis performed in the present work unifies all the extreme-value problems and takes into account all the possible distributions of soil properties, external loads, and pore-water pressure. The analysis is performed without any a priori assumptions regarding $y(x)$ and $\sigma(x)$; all the LE equations are satisfied and no assumptions are used in the investigation process. The most important result of the present work is the basic theorem of LE, which states that in

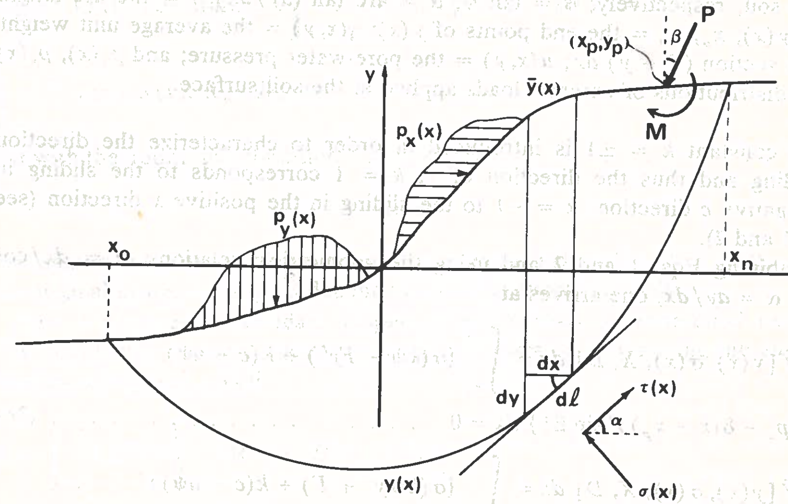

FIG. 2.—Unified Extreme-Value Problem

the rigorous LE treatment the extreme value X_{ex} of X is independent of normal stress distribution $\sigma(x)$.

Mathematical Presentation

A mass of soil (Fig. 2) is considered to be in the state of LE if:

1. Coulomb's relation is satisfied along potential slip line $y(x)$

$$\tau = \frac{c + (\sigma - u)\psi}{F} \quad \ldots \ldots \ldots \quad (1)$$

2. The equations of horizontal, vertical, and moment equilibrium are satisfied for the sliding mass

$$\int_l (k\tau \cos\alpha - \sigma \sin\alpha)\, dl + \int_{x_o}^{x_n} p_x\, dx - P \sin\beta = 0 \quad \ldots \ldots \quad (2a)$$

$$\int_{l} (k\tau \sin\alpha + \sigma\cos\alpha)\, dl - \int_{x_o}^{x_n} [p_y + \hat{\gamma}(\bar{y} - y)]\, dx - P\cos\beta = 0 \quad \ldots \quad (2b)$$

$$\int_{l} [k\tau\cos\alpha - \sigma\sin\alpha)\, y - (k\tau\sin\alpha + \sigma\cos\alpha)\, x]\, dl$$

$$+ \int_{x_o}^{x_n} [p_x\bar{y} + p_y x + \hat{\gamma}(\bar{y} - y)\, x]\, dx + P(x_p\cos\beta - y_p\sin\beta) + M = 0 \quad (2c)$$

in which $c(x,y,y')$ and $\phi(x,y,y')$ = the cohesion and the internal friction angle of the soil, respectively; $\psi = \tan\phi$; $\alpha = \arctan(dy/dx)$; l = the arc length along $y(x)$; x_o, x_n = the end points of $y(x)$; $\hat{\gamma}(x,y)$ = the average unit weight of soil section $(\bar{y} - y)\, dx$; $u(x,y)$ = the pore-water pressure; and $p_x(x)$, $p_y(x)$ = the distributions of external loads applied at the soil surface.

The constant $k = \pm 1$ is introduced in order to characterize the direction of sliding and thus the direction of τ; $k = 1$ corresponds to the sliding in the negative c direction, $k = -1$ to the sliding in the positive x direction (see Figs. 1 and 2).

Combining Eqs. 1 and 2 and using the geometrical relations $dl = dx/\cos\alpha$, $\tan\alpha = dy/dx$, one arrives at

$$\int_{x_o}^{x_n} \bar{H}[y(x), \sigma(x), X, D]\, dx = \int_{x_o}^{x_n} \{\sigma(k\psi - Fy') + k(c - u\psi)$$

$$+ F[p_x - \delta(x - x_p)\, P\sin\beta]\}\, dx = 0 \quad \ldots \quad (3a)$$

$$\int_{x_o}^{x_n} \bar{V}[y(x), \sigma(x), X, D]\, dx = \int_{x_o}^{x_n} \{\sigma(k\psi y' + F) + k(c - u\psi)\, y'$$

$$- F[p_y + \hat{\gamma}(\bar{y} - y) + \delta(x - x_p)\, P\cos\beta]\}\, dx = 0 \quad \ldots \quad (3b)$$

$$\int_{x_o}^{x_n} \bar{M}[y(x), \sigma(x), X, D]\, dx = \int_{x_o}^{x_n} (\sigma\,[(k\psi - Fy')\, y - (k\psi y' + F)\, x]$$

$$+ k(c - u\psi)(y - xy') + F\{p_y x + \hat{\gamma}(\bar{y} - y)\, x + p_x\bar{y} + \delta(x - x_p)[P(x_p\cos\beta$$

$$- y_p\sin\beta) + M]\})\, dx = 0 \quad \ldots \quad (3c)$$

in which δ = Dirac's delta function.

In Eqs. 3, $X[y(x), \sigma(x)]$ is a parameter to be extremized. The value D represents the data of the problem; it includes the known parameters, which characterize the problem, and the given functions $c(x,y,y')$, $\psi(x,y,y')$, $\hat{\gamma}(x,y)$, $u(x,y)$, $\bar{y}(x)$, $p_x(x)$, and $p_y(x)$, which characterize the soil profile. The generalized extreme-value problem of LE can now be stated as follows: find a pair of functions $y(x)$, $\sigma(x)$ that realize the extremum X_{ex} of X subject to the satisfaction of the three LE equations, Eqs. 3.

Variational Analysis

The problem formulated previously is a nonstandard variational problem, since

X may appear in more than one of Eqs. 3. This difficulty has been recently overcome by Baker and Garber (2). Using a number of theorems of variational calculus it was shown that the functions $y(x)$, $\sigma(x)$, realizing the extremum of X in the system of Eqs. 3, are the same as those realizing the extremum of the functional $Z[y(x), \sigma(x)]$. (In Ref. 2 the case of uniform soil profile was considered. However, the possibility of such a reduction of the problem is independent of the character of the soil profile.)

$$Z = \int_{x_o}^{x_n} \bar{M}[y(x), \sigma(x), X_{ex}, D]\, dx \quad \ldots \quad (4a)$$

subject to the constraints $\int_{x_o}^{x_n} \bar{H}[y(x), \sigma(x), X_{ex}, D]\, dx = 0 \quad \ldots \quad (4b)$

$$\int_{x_o}^{x_n} \bar{V}[y(x), \sigma(x), X_{ex}, D]\, dx = 0 \quad \ldots \quad (4c)$$

and with the additional condition $\underset{\substack{y(x)\\ \sigma(x)}}{\mathrm{Extr}}\, Z = 0 \quad \ldots \quad (4d)$

The system of Eqs. 4 depends on the extreme value X_{ex} of X only, while the original system of Eqs. 3 depends on the functional X itself.

Eqs. 4 represent a standard isoperimetric problem of variational calculus. This problem is, in turn, equivalent to the problem of extremizing the auxiliary functional $G[y(x), \sigma(x)]$

$$G = \int_{x_o}^{x_n} g\, dx = \int_{x_o}^{x_n} (\bar{M} + \lambda_1\bar{H} + \lambda_2\bar{V})\, dx$$

$$= \int_{x_o}^{x_n} \{\sigma(x)\, L[y(x), X_{ex}, \lambda_1, \lambda_2, D] + S[y(x), X_{ex}, \lambda_1, \lambda_2, D]\}\, dx \quad (5a)$$

subject to the constraints (Eqs. 4b and 4c) and with the additional condition

$$\underset{\substack{y(x)\\ \sigma(x)}}{\mathrm{Extr}}\, G = 0 \quad \ldots \quad (5b)$$

It can be seen that this condition (Eq. 5b) is equivalent to the previous one (Eq. 4d). The parameters λ_1 and λ_2 appearing in Eq. 5a are Lagrange's undetermined multipliers. The functions L and S are defined as

$$L = [k\psi(\lambda_2 - x) - F(\lambda_1 + y)]\, y' + k\psi(\lambda_1 + y) + F(\lambda_2 - x) \quad \ldots \quad (6a)$$

$$S = k(c - u\psi)[y'(\lambda_2 - x) + (\lambda_1 + y)] - F([p_y + \hat{\gamma}(\bar{y} - y)](\lambda_2 - x)$$

$$-p_x(\lambda_1 + \bar{y}) + \delta(x - x_p)\{P[(\lambda_2 - x_p)\cos\beta + (\lambda_1 + y_p)\sin\beta] - M\}) \quad (6b)$$

The system of Eqs. 5a, 4b, 4c, 5b is the final reduction of the problem. The system of the necessary conditions for an extremum (system of Euler's equations) is

$$\frac{\partial g}{\partial \sigma} - \frac{d}{dx}\left(\frac{\partial g}{\partial \sigma'}\right) = 0 \quad \ldots\ldots\ldots (7a)$$

$$\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = 0 \quad \ldots\ldots\ldots (7b)$$

The function g (Eq. 5a) does not depend on σ'. Therefore, the first Euler's equation, Eq. 7a, simplifies to $\partial g/\partial \sigma = 0$, and, consequently, one arrives at

$$L[Y(x), X_{ex}, \lambda_1, \lambda_2, D] = 0 \quad \ldots\ldots\ldots (8a)$$

or, explicitly

$$Y' = \frac{\dfrac{k\psi}{F}(\lambda_1 + Y) + (\lambda_2 - x)}{(\lambda_1 + Y) - \dfrac{k\psi}{F}(\lambda_2 - x)} \quad \ldots\ldots\ldots (8b)$$

The solution $Y(x, k\psi/F, \lambda_1, \lambda_2, e)$ of this first-order differential equation is a family of extremals (potential slip lines); e is a parameter of integration. The solution to the problem is realized on a member (the critical slip line) of this family.

A slip line $Y(x, k\psi/F, \lambda_1, \lambda_2, e)$ has to satisfy the geometrical boundary conditions: $Y/(x = x_o) = \bar{y}(x_o)$ and $Y/(x = x_n) = \bar{y}(x_n)$. It follows, therefore, that e and x_n may be expressed in terms of λ_1, λ_2, and x_o: $e = e(\lambda_1, \lambda_2, x_o)$; $x_n = x_n(\lambda_1, \lambda_2, x_o)$.

Writing Eq. 5a for the family of extremals $Y(x, k\psi/F, \lambda_1, \lambda_2, x_o)$ and using Eqs. 5b and 8a one gets

$$\underset{\substack{y(x)\\ \sigma(x)}}{\text{Extr}}\, G = \underset{Y(x)}{\text{Extr}} \int_{x_o}^{x_n} S\, dx$$

$$= \underset{\lambda_1, \lambda_2, x_o}{\text{Extr}} \int_{x_o}^{x_n} S[Y(x), X_{ex}, \lambda_1, \lambda_2, x_o, D]\, dx = 0 \quad \ldots\ldots\ldots (9)$$

Thus, using the first Euler's equation, the problem of extremization of the functional G with respect to two functions $y(x)$, $\sigma(x)$ is reduced to the problem of extremization of the functional $\int_{x_o}^{x_n} S\, dx$ with respect to one function, $Y(x)$. However, $Y(x)$ is determined by Eq. 8b as a function of three parameters λ_1, λ_2, and x_o. Therefore, the functional $\int_{x_o}^{x_n} S\, dx$ is in fact a function of the parameters λ_1, λ_2, and x_o, so that the extremization is to be done with respect to these three parameters.

Inspection of the definition of S, Eq. 6b, shows that is always possible to factorize X_{ex} and write Eq. 9 in the following way

$$\underset{\lambda_1, \lambda_2, x_o}{\text{Extr}} \left\{ \int_{x_o}^{x_n} I[Y(x), \lambda_1, \lambda_2, x_o, D]\, dx - X_{ex} \int_{x_o}^{x_n} J[Y(x), \lambda_1, \lambda_2, x_o, D]\, dx \right\} = 0 \quad \ldots\ldots\ldots (10)$$

The form of the functions I and J varies from problem to problem. It can be shown that Eq. 10 is equivalent to

$$X_{ex} = \underset{\lambda_1, \lambda_2, x_o}{\text{Extr}} \; W[Y(x), \lambda_1, \lambda_2, x_o, D] \quad \text{in which} \quad W = \frac{\displaystyle\int_{x_o}^{x_n} I dx}{\displaystyle\int_{x_o}^{x_n} J dx} \quad \ldots (11)$$

The specification of W for different LE problems will be given subsequently.

Basic Theorem

The value X_{ex} in Eq. 11 will represent the solution to the problem if the constraints, Eqs. 4b and 4c, may be satisfied. However, since Eq. 11 is independent of $\sigma(x)$, any two-parametric $\sigma(x, e_1, e_2)$ function may satisfy these two constraints. Therefore, X_{ex}, defined by Eq. 11, actually provides the solution to the problem, and this solution is independent of $\sigma(x)$.

Thus, the analysis performed so far makes possible to formulate the following basic theorem of limiting equilibrium:

> The extreme value X_{ex} of an extremization parameter X is independent of the normal stress distribution, $\sigma(x)$, along the critical slip line $Y(x)$.

This theorem is completely general. It has been established without the specification of the extreme-value problem dealt with, thus it is valid for every such problem.

In the performed analysis no restrictions were imposed on the character of c, $\hat{\gamma}$, u, $\bar{y}$, p_x, and p_y distributions. The $\phi(x,y,y')$ distribution was restricted to continuous functions, since only in this case the first Euler's equation has a form that is common to the whole soil profile. However, later it will be shown that the basic theorem also holds in the case of discontinuous ϕ distributions.

The basic theorem does not contradict the initial presentation, according to which extremization parameter X depends on both functions $y(x)$ and $\sigma(x)$. In general, functional X depends on both functions. However, as the theorem states, in the class of potential slip lines the dependence on $\sigma(x)$ disappears.

It follows from the analysis that every two-parametric function $\sigma(x, e_1, e_2)$ is a solution of the variational problem. On the other hand, the second Euler's equation, Eq. 7b, provides a first-order differential equation, the solution of which, $\sigma(x, e_1)$, depends on one parameter only. These two results do not contradict each other since $\sigma(x, e_1)$ is included in the family $\sigma(x, e_1, e_2)$. Furthermore, since the numerical value of X_{cr} can be established without the actual determination of the σ function, it is not necessary to investigate the σ solutions.

Eqs. 8b and 11 are the main results of the variational analysis. Eq. 8b characterizes the form of the potential slip lines and Eq. 11 defines function W, the extremization of which provides the numerical value of X_{ex}. Eq. 8b depends on one property function $\phi(x,y,y')$ only. This means that the character of the potential slip lines depends on $\phi(x,y,y')$ distribution only. This does not imply that the critical slip line is independent of $c(x,y,y')$, $\hat{\gamma}(x,y)$, $u(x,y)$,

$\bar{y}(x)$, $p_x(x)$, and $p_y(x)$ distributions. These distributions enter the extremization function W and select the critical slip line from the potential ones. It is important to notice that the extremization of W is carried out over three variables (λ_1, λ_2, and x_o) only. Therefore, the solution may be obtained by the conventional trial-and-error technique, which is a common procedure in slope stability problems.

Geometrical Analysis: Modes of Failure

The result stated in the basic theorem is surprising. In order to understand its nature let us consider the differential equation of the potential slip lines

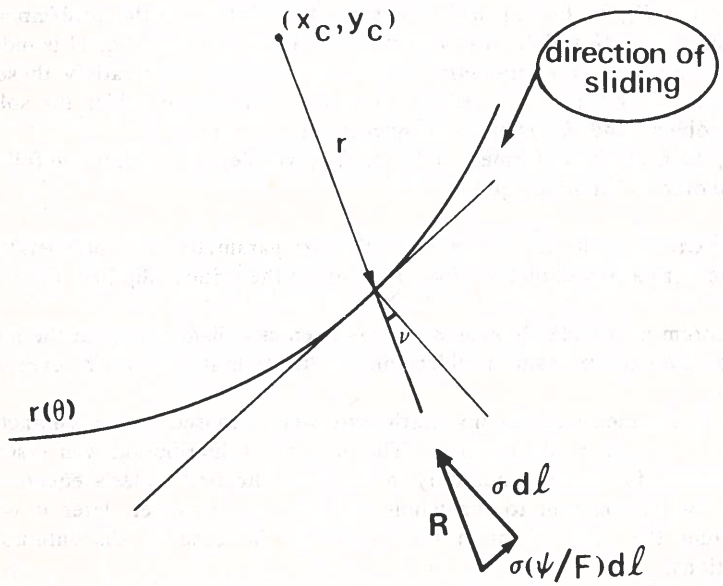

FIG. 3.—Character of Potential Slip Lines

(Eq. 8*b*) in greater detail. Going over to the polar coordinates (r, θ) by means of the following transformation

$$\lambda_2 - x = r(\theta) \cos \theta; \quad \lambda_1 + Y = r(\theta) \sin \theta \quad \ldots \ldots \ldots \ldots \quad (12)$$

Eq. 8*b* may be written as $r'_\theta = \dfrac{k\psi}{F} r$ . . . . (13)

The coordinates transformation implies that parameters λ_1 and λ_2 may be interpreted as coordinates of a point. Thus, in the polar coordinate system with the origin at the point ($x_c = \lambda_2$, $y_c = -\lambda_1$), the family of the potential slip lines is described by Eq. 13.

From Eqs. 12 and 13 and the definition of k it can be checked that in all the problems being considered (Fig. 1), the expected sliding (failure) occurs in the direction of increasing radius vector r (see Fig. 3). It is well known

that in the case of polar curve $r(\theta)$ the acute angle ν between normal to the curve and radius vector is given by

$$\tan \nu = \left| \frac{r'_\theta}{r} \right| \quad \ldots \ldots \ldots \ldots \quad (14)$$

From Eq. 13 it follows that $|r'_\theta / r| = \psi / F$. Thus, for our curves

$$\nu = \text{arc tan} \left(\frac{\psi}{F} \right) \quad \ldots \ldots \ldots \ldots \quad (15)$$

Eq. 15 shows that every point of a potential slip line the acute angle between normal to the line and the radius vector is equal to arc tan (ψ/F).

On the other hand, σ enters the LE equations only by means of vectors of normal forces $\sigma\mathbf{dl}$ and vectors of frictional forces $\sigma(\psi/\mathbf{F})\mathbf{dl}$. Thus (see Fig. 3) at every point of a slip line the resultant vector $\mathbf{R} = \sigma\mathbf{dl} + \sigma(\psi/\mathbf{F})\mathbf{dl}$ is inclined to the normal to slip line by the angle arc tan (ψ/F). Therefore, at every point of a slip line, vector $\mathbf{R}$ coincides with the direction of the radius vector. Thus, $\mathbf{R}$ always passes through the origin (x_c, y_c) of the polar coordinates. This means, that the moment equation written about the point (x_c, y_c) should not include $\sigma(x)$. By writing such an equation one gets $\int_{x_o}^{x_n} S\, dx$. Referring to Eq. 9, we recognize it as a moment equation written about (x_c, y_c). Eq. 11 is the final version of Eq. 9. Thus, we conclude that Eq. 11, providing the solution, is actually the moment equation written about the point (x_c, y_c).

In the cases when at least one of λ_1, λ_2 is infinite, point $(x_c, y_c) \rightarrow \infty$ and the system of vectors $\mathbf{R}$ becomes a system of parallel vectors (passing through a common point at infinity). The common direction of the $\mathbf{R}$ vectors is obviously

$$\tan \mu = \frac{y_c}{x_c} = \frac{-\lambda_1}{\lambda_2} \quad \ldots \ldots \ldots \ldots \quad (16)$$

It follows that there exists a direction $\tan \eta = \lambda_2/\lambda_1$ (perpendicular to the μ direction) for which the equation of equilibrium will be independent of $\sigma(x)$. By writing such an equation one gets the equation $\int_{x_o}^{x_n} S dx = 0$, specified for the cases under consideration. Thus, in these limit cases, Eq. 11 is reduced from being a moment equation to the equilibrium equation, written for the η direction. For the limit cases the equation of the potential slip lines, Eq. 8*b*, is reduced to

$$Y' = \frac{\dfrac{k\psi}{F} \tan \mu - 1}{\tan \mu + \dfrac{k\psi}{F}} \quad \ldots \ldots \ldots \ldots \quad (17)$$

All the limit cases are described by the μ values from the diapason $-\pi/2 < \mu < \pi/2$.

Thus, we have two modes of potential slip lines. The first mode corresponds to finite values of λ_1, λ_2 and is described by Eq. 8*b* or Eq. 13. For this mode, the basic Eq. 11 is a moment equilibrium equation. Therefore, the first mode of slip lines may be considered as corresponding to rotational mode of failure.

The second mode is a limit case of the first one, when at least one of λ_1, λ_2 is infinite. This mode is described by Eq. 17. The basic Eq. 11 is a linear equilibrium equation. Therefore, the second mode of slip lines may be considered as corresponding to translational mode of failure. The analysis performed in the present section shows that the result stated in the basic theorem is a consequence of the geometrical properties of the potential slip lines. Therefore, it is possible to formulate the basic theorem in purely geometrical terms:

> The character of the potential slip lines is such that the system of vectors $\mathbf{R} = \boldsymbol{\sigma}\mathbf{dl} + \boldsymbol{\sigma}(\boldsymbol{\psi}/\mathbf{F})\mathbf{dl}$ either passes through a common point (x_c, y_c) (rotational mode of failure, or it is a system of parallel vectors (translational mode of failure).

The geometrical formulation of the basic theorem explains why extreme value X_{ex} is independent of $\sigma(x)$. Both formulations of the basic theorem are equivalent since either one implies the other. However, the geometrical formulation is more explicit and contains a "recipe" which makes possible the construction of potential slip lines.

Layered Soil Profiles

Eq. 8*b* implies, that the character of the potential slip lines depends only on the ϕ distribution. The distributions c, γ, u, p_x, and p_y enter Eq. 11 under the sign of integration, and therefore their character does not influence the analysis. This means that a special analysis is necessary only in the case of discontinuous ϕ distributions. In practice this case is relevant to a soil profile consisting of a number of layers. For such a profile, ϕ is a partially continuous function with discontinuities at the interfaces.

The analysis presented so far pertains to the case of continuous ϕ distributions. Now the case of discontinuous ϕ distributions will be considered. It can be shown (Bolza, Ref. 3) that in this case: (1) Euler's Eq. 8*b* is valid for each layer; (2) isoperimetric constants λ_1, λ_2 have the same values for every layer; and (3) a curve constructed from segments, which are solutions of Euler's equation for every layer, is a potential slip line. It follows therefore, that the geometrical formulation of the basic theorem is also valid in the case of discontinuous ϕ distributions. Thus, the basic theorem and the basic Eqs. 8*b* and 11 are completely general.

It can be shown that in certain cases a segment of the interface may be a part of the potential slip line. Thus:

> In the case of layered soil profile, the potential slip line consists of segments, which are solutions of the first Euler's equation for every layer. The potential slip line may include also segments of interfaces.

The analysis of the interface conditions is considered to be beyond the scope of the present paper, and it will be given in future publications dealing with numerical solutions of LE problems.

Homogeneous and Isotropic Layers.—For such a profile, solutions of Euler's equation, corresponding to finite values of λ_1 and λ_2 (Eq. 13), are log spirals

$r_i(\theta) = E_i \exp(k\psi_i\theta/F)$ having a common focus (x_c, y_c). Index i is the number of a layer and E_i is the corresponding constant of integration. In the cases when at least one of λ_1, λ_2 is infinite, solutions of Euler's equation (Eq. 17) are straight lines. Therefore:

> In the case of homogeneous and isotropic layers the potential slip line consists of either segments of log-spirals that have a common focus (x_c, y_c) (rotational mode of failure) or segments of straight lines (translational mode of failure). The potential slip line may include also segments of interfaces.

Homogeneous and Isotropic Profile

> In the case of homogeneous and isotropic soil profile the potential slip line is either a log-spiral (rotational mode of failure) or a straight line (translational mode of failure).

Specification of Different Problems

The existence of two distinct modes of failure makes it necessary to write Eq. 11 in the following way

$$X_{ex} = \underset{\lambda_1, \lambda_2, x_o}{\mathrm{Extr}}\; W: \quad (\mathrm{RM}); \quad X_{ex} = \underset{\mu, x_o}{\mathrm{Extr}}\; W: \quad (\mathrm{TM}) \qquad (18)$$

in which RM and TM designate the rotational and translational modes of failure, respectively. Function W was defined as $\int_{x_o}^{x_n} I dx / \int_{x_o}^{x_n} J dx$ (see Eq. 11) which I and J are determined as a result of the factorization of X_{ex} from Eq. 9.

Function S, entering Eq. 9, may be written as

$$S = A - [B + \delta(x - x_p)\, C]\, F \qquad (19)$$

in which $A = k(c - u\psi)[(Y'(\lambda_2 - x) + (\lambda_1 + Y)]: \quad (\mathrm{RM});$

$$A = k(c - u\psi)(Y' - \tan\mu): \quad (\mathrm{TM}) \qquad (20a)$$

$$B = [p_y + \hat{\gamma}(\bar{y} - Y)](\lambda_2 - x) - p_x(\lambda_1 + \bar{y}): \quad (\mathrm{RM});$$

$$B = p_y + \hat{\gamma}(\bar{y} - Y) + p_x \tan\mu: \quad (\mathrm{TM}) \qquad (20b)$$

$$C = P[(\lambda_2 - x_p)\cos\beta + (\lambda_1 + y_p)\sin\beta] - M: \quad (\mathrm{RM});$$

$$C = P(\cos\beta - \tan\mu\,\sin\beta): \quad (\mathrm{TM}) \qquad (20c)$$

Auxiliary functions A, B, and C will be used in the specification of W for different problems.

Slope Stability Problem ($X_{ex} = F$, $k = 1$).—In this case function W has the following form

$$W = \frac{\displaystyle\int_{x_o}^{x_n} A\, dx}{\displaystyle\int_{x_o}^{x_n} B\, dx + C}: \quad (\mathrm{RM\ and\ TM}) \qquad (21)$$

It is of interest to consider the classical problem of a straight slope in a homogeneous and isotropic soil without pore pressure or external loads. As it was established before, in this case the potential slip line is either a log spiral or a straight line. It follows that for the homogeneous case the present approach leads to two solutions. It can be shown that the first one (based on log spirals) is equivalent to the solution procedure suggested by Rendulic (7), while the second one (based on straight lines) is equivalent to Culmann's solution (4).

It is well known that Rendulic's solution yield results that are almost identical with those given by Taylor (8). Culmann's results are generally less conservative than those given in Taylor's stability chart. It follows, therefore, that in the case of homogeneous slope, a solution based on the present approach is practically identical with Taylor's chart.

Limiting Load Problem ($X_{ex} = P$).—In this case function W is given by

$$W = \frac{\int_{x_o}^{x_n} \left(\frac{A}{F} - B\right) dx + M}{(\lambda_2 - x_p)\cos\beta + (\lambda_1 + y_p)\sin\beta} : \quad (\text{RM});$$

$$W = \frac{\int_{x_o}^{x_n} \left(\frac{A}{F} - B\right) dx}{\cos\beta - \tan\mu \sin\beta} : \quad (\text{TM}) \qquad (22)$$

Two types of problems are included in this case, i.e., the bearing capacity of foundation and the limiting forces (active and passive) acting on retaining structures. These two types of problems differ from each other in the assumed nature of the force P. In the bearing capacity problem P is assumed to be an external force, whose location and inclination are known. In the retaining structure, problem P is assumed to be an interaction force whose location and inclination are governed by the possible mode of displacement of the structure under consideration. The LE approach cannot handle displacements; therefore, the distinction between the two types of problems is lost in the present approach. However, the global effect of the possible mode of displacement can be incorporated in the LE analysis by the appropriate choice of the location (x_p, y_p) and inclination β of P. The difference between the active and passive cases is regulated by the parameter k: ($k = 1$) corresponds to the active case; $k = -1$ to the passive case).

In the classical bearing capacity problem (uniform soil and horizontal soil surface) the straight line can not be a slip line. Therefore, log spiral is the critical slip line. The rigorous solution of the problem was obtained by Garber and Baker (5). The variational analysis performed in that work confirmed the applicability of the principle of superposition suggested by Terzaghi (9). It was found that the results agree well with known experimental data.

Limiting Location Problem ($X_{ex} = x_p$).—This case corresponds to the problems of the critical location of force P. Such is, for instance, the problem of the minimal distance from the head of a slope, at which a given load, P, may be safely applied. The segment of the soil surface, on which the load is to be applied, may be approximated by a straight line. Therefore, the relation

between x_p and y_p may be taken as $y_p = ax_p + b$ (a and b are constants). Function W is expressed by

$$W = \frac{\int_{x_o}^{x_n} \left(\frac{A}{F} - B\right) dx - P[\lambda_2 \cos\beta + (\lambda_1 + b)\sin\beta] + M}{P(a\sin\beta - \cos\beta)} : \quad (\text{RM}) \qquad (23)$$

Physically, translation is indifferent to the location of an external force. Therefore, in the present problem only the rotational mode has to be considered. This

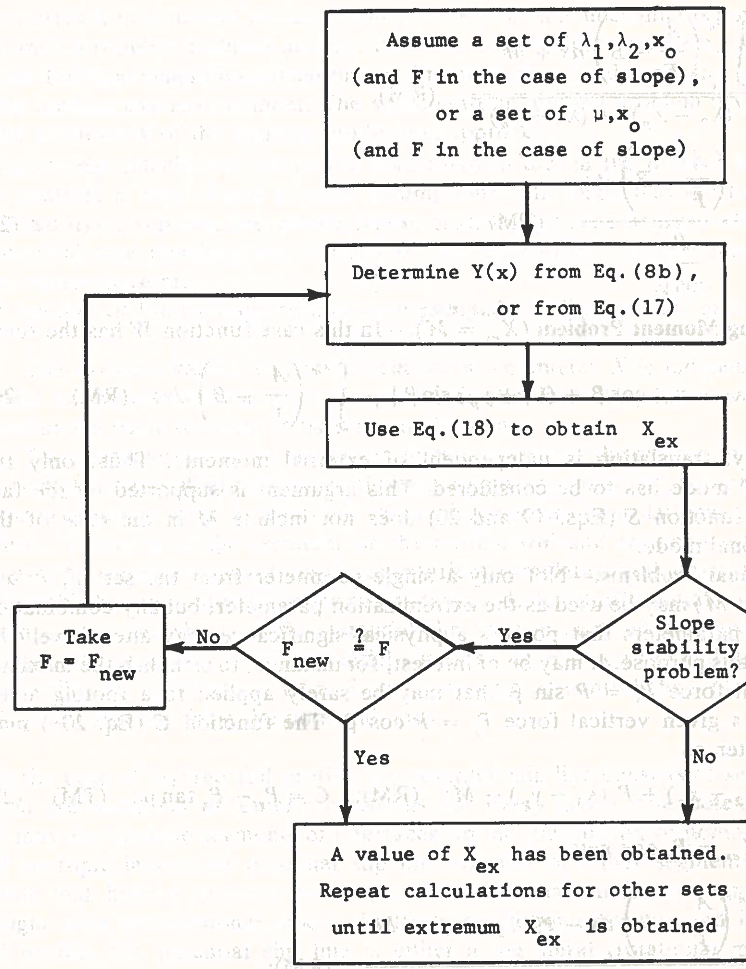

FIG. 4.—Computation Scheme

observation is reflected in the analysis by the fact that the function S (Eqs. 19 and 20) is independent of x_p in the case of translational mode.

Limiting Direction Problem ($X_{ex} = \beta$).—It may be of interest to find the maximum value of β, i.e., to establish the critical inclination at which a given load, P, may be safely applied. In order to factorize parameter β, function C (Eq. 20c) has to be rewritten

$$C = P\sqrt{(\lambda_2 - x_p)^2 + (\lambda_1 + y_p)^2}\cos(\beta + \omega) - M: \quad \text{(RM)};$$

$$C = P\frac{\cos(\beta + \mu)}{\cos\mu}: \quad \text{(TM)} \quad \text{(24)}$$

in which $\omega = \arctan[(\lambda_1 + y_p)/(x_p - \lambda_2)]$. Taking $\cos(\beta + \omega)$ or $\cos(\beta + \mu)$ as X_{ex}, the function W is given by

$$W = \frac{\int_{x_o}^{x_n}\left(\frac{A}{F} - B\right)dx + M}{P\sqrt{(\lambda_2 - x_p)^2 + (\lambda_1 + y_p)^2}}: \quad \text{(RM)};$$

$$W = \frac{\int_{x_o}^{x_n}\left(\frac{A}{F} - B\right)dx}{\frac{P}{\cos\mu}}: \quad \text{(TM)} \quad \text{(25)}$$

Limiting Moment Problem ($X_{ex} = M$).—In this case function W has the form

$$W = P[(\lambda_2 - x_p)\cos\beta + (\lambda_1 + y_p)\sin\beta] - \int_{x_o}^{x_n}\left(\frac{A}{F} - B\right)dx: \quad \text{(RM)} \quad \text{(26)}$$

Physically, translation is independent of external moments. Thus, only the rotational mode has to be considered. This argument is supported by the fact that the function S (Eqs. 19 and 20) does not include M in the case of the translational mode.

Additional Problems.—Not only a single parameter from the set (F, P, x_p, y_p, β, and M) may be used as the extremization parameter, but any combination of these parameters that possess a physical significance may alternatively be used for this purpose. It may be of interest, for instance, to establish the maximal horizontal force $P_x = P\sin\beta$ that may be safely applied to a footing acted upon by a given vertical force $P_y = P\cos\beta$. The function C (Eq. 20c) may be rewritten as

$$C = P_y(\lambda_2 - x_p) + P_x(\lambda_1 + y_p) - M: \quad \text{(RM)}; \quad C = P_y - P_x\tan\mu: \quad \text{(TM)} \quad \text{(27)}$$

Taking $X_{ex} = P_x$ one gets

$$W = \frac{\int_{x_o}^{x_n}\left(\frac{A}{F} - B\right)dx - P_y(\lambda_2 - x_p) + M}{\lambda_1 + y_p}: \quad \text{(RM)};$$

$$W = \frac{P_y - \int_{x_o}^{x_n}\left(\frac{A}{F} - B\right)dx}{\tan\mu}: \quad \text{(TM)} \quad \text{(28)}$$

Computation Scheme

The computation scheme for an extreme-value problem is given in Fig. 4. As it is seen from the scheme, in the case of the slope-stability problem an additional iterational procedure in F is needed. The necessity of this procedure follows from the fact that F enters the equations of potential slip lines (Eqs. 8b and 17).

Summary and Conclusions

A unified formulation of extreme-value problems of soil mechanics is presented. The unified problem includes among others the slope-stability problem, problems of the limiting magnitude, location or direction of a load, and the problem of the limiting external moment. The solution to the unified problem is obtained in the framework of the limiting equilibrium approach.

Variational calculus served as the mathematical tool in the present analysis. The analysis is free of any a priori assumptions with respect to the functions $y(x)$ and $\sigma(x)$ (slip line and normal stress distribution). The analysis takes into account all the possible distributions of soil properties, external loads, and pore-water pressure.

A number of fundamental results were established. These results are:

1. The extreme value, X_{ex}, of extremization parameter X is independent of normal stress distribution $\sigma(x)$ along critical slip line $Y(x)$. This statement constitutes a basic theorem of limiting equilibrium.

2. The character of the potential slip lines is such that the system of vectors $\mathbf{R} = \sigma\mathbf{dl} + \sigma(\psi/\mathbf{F})\mathbf{dl}$ either passes through a common point (x_c, y_c) (rotational mode of failure), or it is a system of parallel vectors (translational mode of failure). Vector $\mathbf{R}$ is the resultant of the normal $\sigma\mathbf{dl}$ and frictional $\sigma(\psi/\mathbf{F})\mathbf{dl}$ forces. This statement constitutes the geometrical formulation of the basic theorem and establishes two alternative modes of failure. Euler's differential equations, which control the shape of the potential slip lines in each of two modes, are derived. The character of the potential slip lines depends on the ϕ distribution only while the distributions of c, $\hat{\gamma}$, u, $\bar{y}$, p_x, and p_y control the selection of the critical slip line from the potential ones.

In the case of layered soil profile, the potential slip line consists of segments, which are solutions of Euler's equations for every layer. The potential slip line may also include segments of interfaces. In the special case of homogeneous and isotropic layers the potential slip line consists of either segments of log spirals that have a common focus (x_c, y_c) (rotational mode) or segments of straight lines (translational mode). In the case of homogeneous and isotropic soil profile, the potential slip line is either a log spiral (rotational mode) or a straight line (translational mode).

The critical value of an extremization parameter follows from the extremization of a function W, the form of which varies from one particular problem to another. The explicit form of this function, corresponding to different problems, is presented. The extremization of W is carried out over three variables in the case of rotational mode and over two variables in the case of translational mode. Thus, a simple trial and error procedure may be used for the determination of numerical results.

The approach presented herein has been applied to two classical problems: (1) Bearing capacity of shallow foundations; and (2) the stability of straight slopes. Comparison of the results obtained by the present approach, with available solutions, confirms the validity of the generalized analysis. It seems, therefore, that the present approach may be applied with confidence to the analysis of different extreme-value problems, for which no reliable solutions are presently available.

Acknowledgment

The writers express their gratitude to M. M. Schiffer of Stanford University, Faculty of Mathematics, Stanford, Calif., for his interest in the present work, useful conversation, and a number of important notes.

Appendix I.—References

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Appendix II.—Notation

The following symbols are used in this paper:

A, B, C = auxiliary functions;
a, b = parameters;
c = cohesion;
D = data of problem;
E_i, e, e_1, e_2 = parameters of integration;
F = safety factor with respect to strength;
G = auxiliary functional;
g = Lagrange's function;
$\bar{H}, \bar{V}, \bar{M}$ = auxiliary functions;

I, J = auxiliary functions;
i = number of layer;
k = auxiliary parameter;
L, S = auxiliary functions;
l = arc length along $y(x)$;
M = external moment;
P = external force;
P_x, P_y = projections of P;
p_x, p_y = distributions of external loads;
$\mathbf{R}$ = resultant vector;
$r(\theta)$ = potential slip line in polar coordinates;
u = pore-water pressure;
W = extremization function;
X = extremization functional;
X_{ex} = extreme value of X;
$x_c = \lambda_2$ = origin of polar coordinates;
x_o, x_n = end points of $y(x)$;
x_p, y_p = point of P application;
$Y(x)$ = potential slip line (extremal);
$y_c = -\lambda_1$ = origin of polar coordinates;
$y(x)$ = slip line (general);
$\bar{y}(x)$ = soil surface;
Z = auxiliary functional;
α = angle of $y(x)$ inclination;
β = direction of P;
$\hat{\gamma}$ = average unit weight of soil section $(\bar{y} - y)\,dx$;
δ = Dirac's function;
η = direction perpendicular to μ direction;
λ_1, λ_2 = Lagrange's undetermined multipliers;
μ = direction of $\mathbf{R}$ (translational mode of failure);
ν = angle between normal to $r(\theta)$ and radius vector;
$\sigma(x), \tau(x)$ = normal and tangential stress distributions;
ϕ = angle of internal friction;
ψ = tan ϕ; and
ω = auxiliary angle.

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TECHNICAL NOTES

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DISCUSSION

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[a]Discussion period closed for this paper. Any other discussion received during this discussion period will be published in subsequent Journals.

16000 PILE LOAD TESTS: CYCLIC LOADS AND LOAD RATES

KEY WORDS: Capacity; Clays; Compression; **Cyclic loads; Loading rate; Pile load tests (cyclic loading); Soils; Steel piles;** Stratigraphy; Tension; **Unconsolidated soils**

ABSTRACT: Two series of axial load tests were performed on four 14-in. (356-mm) diam, open-end, steel pipe piles at an interval of about 320 days. Pile lengths of 40 ft or 50 ft (12.2 or 15.2 m) were installed below conductors driven to depths ranging from 115 ft to 320 ft (35 m to 98 m) into a strong underconsolidated clay. Each pile was subjected to as many as 26 tests. Data presented includes: (1) Compression and tension tests; (2) tests performed at different times after driving and after previous tests; and (3) incremental loading and constant rate of loading. Pile capacity increased 40% to 75% when the loading rate increased by three orders of magnitude. The one-way cyclic loading applied in this study did not effect the ultimate capacity, but large displacement began to accumulate when the maximum cyclic load reached 80% of the static capacity.

REFERENCE: Kraft, Leland M., Cox, William R., and Verner, Edward A., "Pile Load Tests: Cyclic Loads and Varying Load Rates," *Journal of the Geotechnical Engineering Division,* ASCE, Vol. 107, No. GT1, **Proc. Paper 16000**, January, 1981, pp. 1-19

15980 LONGITUDINAL VIBRATIONS OF EMBANKMENT DAMS

KEY WORDS: **Earth dams; Earthquakes; Embankments;** Soil dynamics

ABSTRACT: This paper presents an analytical method to study free and forced longitudinal vibrations of embankment dams. Both shear and dilatational deformations are taken into account, and the dam is modeled as a linear homogeneous prism with a wedge-shape cross section, bounded by two vertical (abutment) and one horizontal (riverbed) planes. Numerical results demonstrate the effect of the length and height of the dam on its natural frequencies and its modal displacement and strain shapes. It is shown that shear deformations are more important in relatively long dams, whereas the opposite is true with dams built in narrow canyons. The method is evaluted through two case histories involving an earthfill and a rockfill dam. Close agreement exists between predicted and observed natural frequencies, but the distribution of peak accelerations within the dam is badly predicted.

REFERENCE: Gazetas, George, "Longitudinal Vibrations of Embankment Dams," *Journal of the Geotechnical Engineering Division,* ASCE, Vol. 107, No. GT1, **Proc. Paper 15980**, January, 1981, pp. 21-40

15979 CYCLIC AXIAL RESPONSE OF SINGLE FILE

KEY WORDS: **Bearing capacity;** Consolidation; **Cyclic loads;** Foundations; **Offshore structures; Pile settlement;** Piles (foundations); **Shear strain;** Skin friction; Soil mechanics

ABSTRACT: A number of solutions have been obtained for a typical offshore pile, to determine the influence of a number of the input parameters on the computed cyclic response. More significant parameters include the critical shear strain at which significant degradation of skin friction occurs, the rate of loading on the pile, and the distributions of static skin friction and soil modulus along the pile. The analysis predicts a gradual decrease in cyclic stiffness of the pile with increasing numbers of cycles and increasing cyclic load level, but a very sudden decrease in ultimate load capacity once the (half-peak-to-peak) cyclic load level exceeds 50% to 60% of the ultimate static load capacity. These theoretical findings are broadly confirmed by the results of small-scale laboratory model tests.

REFERENCE: Poulos, Harry G., "Cyclic Axial Response of Single File," *Journal of the Geotechnical Engineering Division,* ASCE, Vol. 107, No. GT1, **Proc. Paper 15979**, January, 1981, pp. 41-58

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Extreme-Value Problems of Limiting Equilibrium[a]

Discussion by Reginald A. Barron,[3] F. ASCE

This paper presents an analysis method for stability problems at limiting equilibrium using the method of variational calculus. This calculus is not new, but the writer doubts that many engineers in the field are familiar with it. If it is to be used by the profession, additional details should be given. It is considered by the writer that important omissions were made in this paper which, had they been included, would have improved the clarity and value of the paper. These omissions are:

1. No list of assumptions is given. Near the end of the introduction the authors state that none are used. Certainly the use of the effective stress concept as expressed by Eq. 1 is one. Another is the use of safety factor, F, to apply not only for the overall stability, but also as a point safety factor when F is greater than unity.
2. No illustrative examples are given. To aid the comprehension by most engineers a detailed example would be most useful.
3. No comparison with results of others are given. Near the end of the section on "Slope Stability Problems" the authors state the results of their method "are almost identical with those given by Taylor (8)." If this is so, what is the purpose for using a more complicated method of analysis?
4. No discussion is included indicating the influence of the soil stress-strain properties on the validity of the method. In fact, in the second paragraph of the "Summary and Conclusions" the authors state that the method "takes into account all the possible distribution of soil properties . . . and pore-water pressures."

Nowhere in the paper are the strength parameters defined. The writer assumes since an effective stress equation is used (Eq. 1) that the strength parameters are those obtained from consolidated-drained (S) shear tests. The writer concurs in the application of Eq. 1 to failure conditions where F is equal to unity; but for cases where F is greater than unity the drained shear strength for the nonfailure effective normal stress is generally not available. See Barron (10,11) Gould (14) and Johnson (12,13). Except for dry, cohesionless soils the potential shear strength of a wet soil not at a failure condition is the undrained strength that can be obtained without any change in the water content. The shear strength tests are, therefore, the consolidated-undrained (R) or the unconsolidated-undrained (Q) as the conditions to be analyzed indicate. In the writer's opinion,

[a] October, 1979, by Michael Garber and Rafael Baker (Proc. Paper 14901).

[3] Consulting Engr., 62 Horshoe Rd., Guilford, Conn. 06437.

the concept indicated by Eq. 1 is the most misused idea of soil mechanics.

If a stable stress condition exists in a moist soil and an undrained shear test is made, the effective stress path for a dilative soil has a positive slope and at failure the pore pressure can be negative (gage) resulting in a shear strength at failure larger than that given by Eq. 1 with the effective normal stress that which exists on the potential failure surface at the initiative of the test. Because large negative pore pressures (gage) can occur in a test using large back pressure to prevent pore-water cavitation, the full shear strength given by this test may not be conservative to use in a stability study.

If the soil is compressive (strain-softening), then the effective stress path rises to the failure strength curve with a negative slope. Large positive pore pressures can exist and the shear strength at failure is less than that indicated by Eq. 1. The shear strength indicated by Eq. 1 can be realized only if water content changes can occur. If they cannot, then it is improper to use the results of Eq. 1.

In addition, if F, the average safety factor, is larger than unity, then the stress conditions in the soil mass are highly indeterminate and the concept of a limiting equilibrium condition with a reduced shear strength being applied to a given problem is certainly an assumption, but one which is commonly made without, however, claims being made for any greater degree of exactness. An excellent discussion of the difficulties of analyzing slopes is given by Johnson (12,13). It would be desirable if the authors define Dirac's function and Lagrange's undetermined multipliers.

In conclusion, the writer questions the advisability of publishing theoretical papers of this kind without it being so written that it can be comprehended by a large portion of the society's membership.

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Discussion by Ryszard J. Izbicki[4]

The authors applied a generalized limit equilibrium approach to the problems of slope stability and problems of limiting magnitude, and location or direction of a load and external moment. The most important result is a basic theorem

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of limit equilibrium which states that the critical value of the extermization parameter (F = the factor of safety; P = an external load; M = an external moment, etc.) is independent of the normal stress distribution along the critical slip line. Simultaneously, the variational analysis establishes the existence of two (rotational and translational) modes of failure mechanism. In the case of the rotational mode of failure the potential slip lines are a log-spiral but if the translational mode of failure exists, the potential slip lines are straight lines. The analysis was performed without any a priori assumptions regarding slip line and normal stress distribution; all the equilibrium equations are satisfied and no aditional assumptions were in the investigation process. The results stated in the paper are very interesting but are not surprising.

So, approximate methods for solving the problems of soil stability may be divided into (15,16,18): (1) Methods of limit analysis based on two limit theorems; and (2) methods of approximate satisfying of conditions of equilibrium and yield criterion only in the definite points or regions of the material considered. These method are called the limit equilibrium methods.

In the limit analysis for elastic or rigid-plastic material obeying the associated flow law the validity of two basic limit theorems is proved, making use of the principle of maximum plastic work. According to the statement of this theorems, in order to properly bound the "true" solution, it is necessary to find a kinematically admissible failure mechanism (velocity or flow field) in order to obtain an upper bound solution. A statically admissible stress field, satisfying all the equilibrium conditions and nowhere violating the yield criterion, will be required for a lower bound solution. If the upper and lower bounds coincide, the exact value of the collapse, or limit, load (factor of safety, etc.) is determined (15,18).

An upper bound solution may be obtained (see Ref. 18) by: (1) Comparing the sum of the total work of external forces and total work of body forces with the total internal dissipation of energy (this method is called herein energy approach); or (2) making use of equilibrium conditions of the field of forces (stresses) associated with the assumed kinematically admissible failure mechanism (equilibrium approach).

Both the approaches are equivalent. It can be observed, that the equations of the work balance, may be treated as an equation of virtual work and, thus, as an equation that expresses the condition of global equilibrium. A number of examples solved by means of the two approaches are discussed in Ref. 18.

The upper bound equilibrium approach is more advantageous in relation to the analysis starting from work balance. Namely, it provides more information about the stress distribution inside the material or the force (load) distribution along contact lines. There are, however, difficulties in applying this approach in the case of layered soil profile and composed failure mechanisms, which consist of rigid zones (blocks) and soft zones.

In the case of associated flow law the energy dissipated within the failure mechanism is independent of the stress distribution. Therefore the determined value of limit load (and all other parameters) will be independent of the internal stress distribution. In the kinematically admissible failure mechanism, the potential slip lines are either log-spirals (rotational mode) or straight lines (translational mode).

The aforementioned properties of the upper bound solutions and kinematically admissible mechanisms follow from the associated flow law. The same properties are obtained by the authors by applying, in the light of the assumed classification of methods, the upper bound generalized equilibrium approach. Their considerations, however, are limited to the case of failure mechanisms consisting of one rigid block which undergoes either rotation or translation.

The upper bound energy approach allows the consideration of complex failure mechanisms composed of a number of rigid blocks which undergo either translation or rotation, as well as of mechanisms containing soft zones, which are deformed during plastic strain. Some examples on composed failure mechanisms have been given by Gudehus (17), Chen (15), the writer and Mróz (18), and Karal (19).

To conclude we may add that conventional limit equilibrium methods based on static equilibrium of forces (or stresses) acting on the yield region are in many cases equivalent to the upper bound method (see equilibrium approach). However, the assumed failure mechanism must be kinematically admissible and it must be associated by the appropriate volumetric dilatation. When the failure mechanism is not kinematically admisible from viewpoint of limit analysis, it may be neither upper bound not the lower one. However, each upper bound (equilibrium approach) solution is always equivalent to the limit equilibrium method (15,18).

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Discussion by A. Luceño[5] and E. Castillo[6]

The authors state a generalized extreme-value problem of *LE* as a search for a pair of functions $y(x)$ and $\sigma(x)$ that realize the extremum X_{ex} of X subject to the satisfaction of the three LE equations (3).

However it can be easily demonstrated that functional $X[y(x),\sigma(x)]$ is, in general, unbounded. Therefore there exists a potential sliding line $y(x)$ and a normal stress distribution $\sigma(x)$ giving a safety factor as small as desired,

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and in consequence the extreme problem is incorrectly stated.

In order to give an example of the unbounded character of functional $X[y(x),\sigma(x)]$ let us consider Eqs. 3*a*, 3*b* and 3*c* for the case of a slope in an homogeneous soil and lack of pore water pressure and external loads. These equations in adimensional form become (when $k = 1$):

$$\int_{X_0}^{X_1} [(N + S\psi) - FSY'] \, dX = 0 \quad \text{(29)}$$

$$\int_{X_0}^{X_1} [(N + S\psi)\, Y' - F(\bar{Y} - Y - S)] \, dX = 0 \quad \text{(30)}$$

$$\int_{X_0}^{X_1} \{(N + S\psi)(Y - Y'X) - F[S(X + Y'Y) - X(\bar{Y} - Y)]\} \, dX = 0 \quad \text{(31)}$$

in which the following adimensional parameters have been used: $N = c/\gamma H$, $S = \sigma/\gamma H$, $\psi = tg(\phi)$, $X = x/H$, $Y = y/H$, $\bar{Y} = \bar{y}/H$, $Y' = dY/dX$, where X_0, X_1 = the abscissas of the end points of the sliding line; and H = the height of the slope.

By making $X = F$ (Baker and Garber, 1978), Eq. 29 gives:

$$F = \frac{\displaystyle\int_{X_0}^{X_1} (N + S\psi) \, dX}{\displaystyle\int_{X_0}^{X_1} S\,Y' \, dX} \quad \text{(32)}$$

So, according to Garber and Baker, the problem can be stated as minimizing Eq. 32 subject to conditions described by Eqs. 30 and 31.

One way of demonstrating that Eq. 32, subject to Eqs. 30 and 31, does not attain an absolute minimum is by selecting a sliding line $Y^*(X)$, not a log-spiral, and by using the Ritz method to check that the functional $F^*[S(X)]$ obtained by substituting the equation of this sliding line in Eq. 32 and Eqs. 30 and 31 has no absolute minimum.

As it is well known, Ritz's method consists of a discretization of the vector space of possible solutions. So, a system of basic functions $[l_i(X)]$ is selected and the solution is assumed to be of the type:

$$S(X) = \sum_{i=1}^{n} \alpha_i l_i(X); \quad \alpha_i = cte, \quad (i = 1, 2, \ldots, n) \quad \text{(33)}$$

According to this assumption the functional $F^*[S(X)]$ becomes a function of n-2 variables from which its minimum must be investigated.

In effect, let us consider a slope defined by the following geometrical and geotechnical characteristics: $c = 1\ Tn/m^2$; $\psi = tg(\phi) = 1$; $\gamma = 2\ Tn/m^3$; and $H = 3\ m$.

The characteristics of the slope, in adimensional form are: $N = c/\gamma H = 1/6$, $\psi = 1$ and its equation is given by:

$$\bar{Y} = 0; \quad \forall X \le 0; \quad \bar{Y} = 3X; \quad \forall X \in (0, 1/3);$$
$$\bar{Y} = 1; \quad \forall X \ge 1/3 \quad \text{(34)}$$

The safety factor of this slope according to the solution given by Euler's equations is approx 1.8.

If the following potential slip line: $Y^*(X) = 3X^2 - 1/3$ and the system of basic functions: $l_1(X) = X^2$; $l_2(X) = X$; $l_3(X) = 1$ are selected, functional $F^*[S(X)]$, according to Eqs. 32, 30 and 31, becomes a function of one variable defined by the following system of equations:

$$(2 - 5F^*)\alpha_1 + (3 - 12F^*)\alpha_2 + (18 - 18F^*)\alpha_3 = -3 \quad \text{(35)}$$

$$(5 + 2F^*)\alpha_1 + (12 + 3F^*)\alpha_2 + (18 + 18F^*)\alpha_3 = -3 + 9F^* \quad \text{(36)}$$

$$(192 + 345F^*)\alpha_1 + (315 + 612F^*)\alpha_2 + (1{,}080 + 1{,}080F^*)\alpha_3$$
$$= -180 + 195F^* \quad \text{(37)}$$

This system can be solved in α_1, α_2, and α_3 for values of F^* not equal to $-1/3$. This shows that Eq. 32 subject to conditions described in Eqs. 30 and 31 is unbounded.

Nevertheless, it is worthwhile to make some comments:

1. Though the safety factor must be positive, this fact is not explicitly reflected in the statement of the problem, and as a consequence the system, Eqs. 35, 36 and 37 can be solved for negative values of F^*.

2. In the same way, the condition given by the Mohr-Coulomb failure criterion:

$$S(X) \geq -N/\psi; \quad \forall X \in (X_0, X_1) \quad \text{(38)}$$

is not included, and as a consequence the system of Eqs. 35, 36, and 37 gives solutions not satisfying this condition.

One way of taking into account these constraints in the statement of the problem is by making:

$$F = K^2 \quad \text{(39)}$$

$$S(X) = [\epsilon(X)]^2 - \frac{N}{\psi} \quad \text{(40)}$$

which lead to the system:

$$\int_{X_0}^{X_1} \left[\psi\epsilon^2 - K^2\left(\epsilon^2 - \frac{N}{\psi}\right) Y'\right] dX = 0 \quad \text{(41)}$$

$$\int_{X_0}^{X_1} \left[\psi\epsilon^2 Y' - K^2\left(\bar{Y} - Y - \epsilon^2 + \frac{N}{\psi}\right)\right] dX = 0 \quad \text{(42)}$$

$$\int_{X_0}^{X_1} \left\{\psi\epsilon^2(Y - Y'X) - K^2\left[\left(\epsilon^2 - \frac{N}{\psi}\right)(X + Y'Y)\right.\right.$$
$$\left.\left. - X(\bar{Y} - Y)\right]\right\} dX = 0 \quad \text{(43)}$$

Eq. 41 defines functional $K^2 = F$ which must be minimized subject to the conditions described by Eqs. 42 and 43.

The extremals of this new problem are exactly the same as those of the old functional and it can be demonstrated by a counterexample that the new functional does not attain a relative minimum either (the first writer, 1979; the writers, 1980).

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Closure by Michael Garber[7] and Rafael Baker[8]

The writers thank all the discussers for the interest expressed in the paper.

The discussion by A. Luceno and E. Castillo suggests that in the extreme-value problem of LE presented by the writers the extremization parameter {functional $X[y(x), \sigma(x)]$} is, in general, unbounded. The discussers claim that there always exist a pair of functions $y(x)$ and $\sigma(x)$ "giving (a) safety factor as small as desired, and in consequence the extreme problem is incorrectly stated." To prove it, the discussers provide a "counter example" that, in their understanding, clearly disqualifies the writers' presentation. The example refers to a slope which, according to the writers' solution, has a safety factor of 1.8. The discussers chose at random a potential slip line $y(x)$ and assumed the pressure distribution $\sigma(x)$ to be presented by a polynomial with three unknown parameters σ_1, α_2, and α_3. Substituting their functions $y(x)$ and $\sigma(x)$ into the writers' equilibrium equations the discussers derived a system of three linear equations with respect to three unknowns α_1, α_2, and α_3. This system can be solved for almost any value of the safety factor, F. It seems therefore, that the discussers can always point out a pair, $y(x)$ and $\sigma(x)$, that not only gives F less than 1.8 but, moreover, F "as small as desired."

The point is, however, that there are natural restrictions on the value of F as well as on the character of the $\sigma(x)$ function, and the discussers recognize this as they make an attempt to modify the problem presentation. These restrictions are: (1) $F \geq 0$ (the safety factor is non-negative by definition); and (2) $\sigma(x) \geq -c/\psi$ while $\sigma \not\equiv -c/\psi$ (a condition supplementing the Mohr-Coulomb failure condition).

These restrictions were not mentioned explicitly in the paper, but as the discussers know, their incorporation does not change the variational solution. If we consider the discussers "counter example" in view of these restrictions,

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then it fails to prove anything but the validity of the writers solution. The nondimensional pressure distribution $S(x)$ assumed by the discussers was inspected in the range $0 \le F \le 4$ [Figs. 5(*b*) and 5(*c*)] against restriction 2, which in nondimensional terms is rewritten as $S(x) \ge N/\psi$. It was found that the minimal S always occurs at the end points x_0 and x_1 [Figs. 5(*a*) and 5(*b*)] of the slip line $y(x)$. As it can be seen from Figs. 5(*b*) and 5(*c*), $S(x)$ satisfies restriction 2 for $F \ge 4$ but fails to satisfy it for any $0 \le F < 4$. This implies that $F = 4$ is the minimum the discussers can claim for their example without violating the obvious restrictions previously mentioned. As 4 is greater than 1.8 the "counter example" clearly fails. Possibly in expectation of this the discussers refer to another example that is supposed to demonstrate their point. The reference is to two papers in Spanish (local bulletins) and one in English

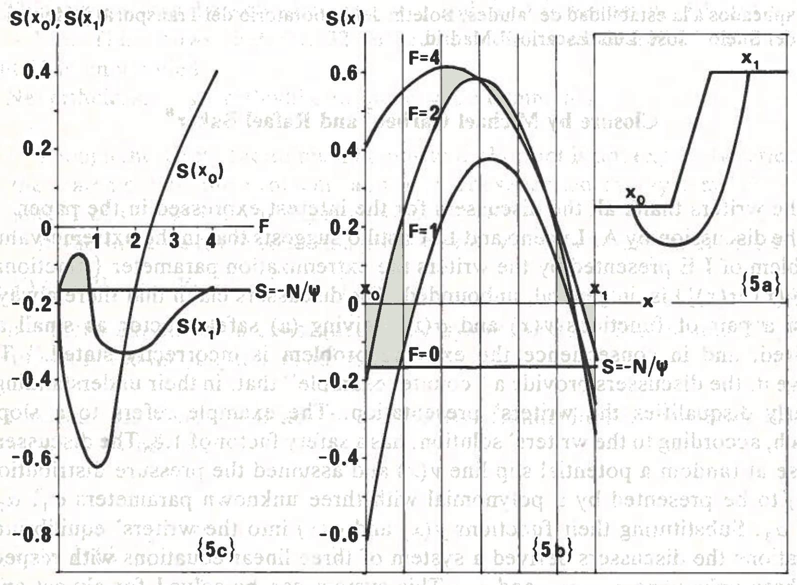

FIG. 5.—Investigation of Discusser's Example: (*a*) Discusser's Potential Slip Line; (*b*) Pressure Distribution $S(x)$ for Range $0 < F < 4$; (*c*) Pressure at End Points x_0 and x_1 for Range $0 < F < 4$

(Symposium in India). The writers' efforts to obtain these papers have been unsuccessful and thus comment is reserved.

The variational solutions proved to satisfy restrictions 1 (which in general has to be stated as $X \ge 0$, in which X = the extremization parameter) and 2 as demonstrated by solutions presented by the writers (5). Taking the opportunity, the writers would like to introduce an additional (kinematic) restriction that must be imposed on the shape of $y(x)$. This third restriction states that the sliding is possible only when the curvature of $y(x)$ decreases in the direction of sliding. Thus, there are "proper" and "improper" potential slip lines, and $y(x)$ used by the discussers [see Fig. 5(*a*)] obviously belongs to the class of "improper." The variational (extremal) $Y(x)$ always satisfies the third restriction. Indeed, using Eq. 8*b* the curvature ρ of $Y(x)$ is expressed as

$$\rho = \left| \frac{Y''}{(1 + Y'^2)^{3/2}} \right| = \frac{1}{r\sqrt{1 + \left(\frac{k\psi}{F}\right)^2}} \qquad (44)$$

As was shown in the paper, the radius vector **r** increases in the direction of sliding (see Geometrical Analysis section), consequently ρ, (Eq. 44), decreases in the direction of sliding.

The discussion by R. A. Barron questions the manner in which the material is presented. The writers readily agree with a number of points raised. The original text was about 25% longer and it included more detail explanations about the essence of the method and its limitations. We had, however, to follow the reviewers request to considerably shorten the paper.

The discussion by R. J. Izbicki provides a profound analysis of the writers method from the point of view of the limit analysis approach. The writers found a special interest in this discussion.

Errata.—The following corrections should be made to the original paper:

Page 1158, paragraph 1, line 3: Should read "the negative *x* direction," instead of "the negative *c* direction,"
Page 1164, line 10: Should read "(rotational mode of failure)," instead of "(rotational mode of failure,"