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DISCUSSION

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Probabilistic Model of Progressive Failure of Slopes

By Robin N. Chowdhury,[1] and Dimitri A-Grivas,[2] Members, ASCE

Abstract: A probabilistic model for the progression of failure in a soil slope is presented. Failure progression is defined as a spatial and continuous extension of the failure zone along a potential slip surface in a statistically homogeneous medium. The local safety margin of any segment of the slip surface is assumed to follow a normal distribution; cohesive and frictional parameters of shear strength being considered as independent random variables. The joint distribution of the safety margin of any two adjacent segments of the slip surface is assumed to be bivariate normal. After defining the model and outlining the rules of transition, expressions for the probability of failure progression are derived. The model and its formulation are illustrated by a worked example and the significance of the proposed model is discussed. The suggested approach to the study of progressive failure gives insight into the interdependence of the stability of adjacent elements or sections of a soil mass. Consequently, it is potentially valuable in clarifying the real behavior of soil masses and especially slopes.

Introduction

The progressive character of earth slope failures has been recognized for many decades (11,30). External observations of stable and failed slopes and other research studies have confirmed the belief of geotechnical engineers that failure always starts at one location and then spreads progressively to other regions within a soil mass. Even when there are no external signs of failure, certain regions with a sloping ground may already be overstressed, and propagation of failure may have begun.

Factors that contribute to progressive failure of earth slopes include: (1) Nonuniform stress and strain distributions; (2) strain-softening behavior of soils; (3) stress release; (4) softening (decrease of shear strength) due to the presence of fissures and the action of water; (5) the presence of joints and discontinuities; (6) increase of pore water pressures; (7) minor geological details, e.g., weak lenses of sand or soft clay, and; (8) environmental effects. Whatever the specific nature of each influencing factor, to account for the progressive mechanism of

[1]Reader, Dept. of Civ. Engrg., Univ. of Wollongong, New South Wales, 2500, Australia.
[2]Assoc. Prof., Rensselaer Polytechnic Inst., Troy, N.Y. 12181.
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slope failure is important for both fundamental and practical reasons. Time effects are also important as propagation of failure in space is always accompanied by progression in time. Various aspects of progressive failure have been explored in the literature of Terzaghi (30,31), Skempton (27,28,29), Bishop (6,7), Bjerrum (8), Peck (22), Seed (25), Wilson (34), and Lo and Lee (18).

Deterministic and Probabilistic Methods of Stability Analysis

Deterministic methods of slope stability analysis have been widely used in geotechnical practice. Based on the assumption of limit equilibrium, these methods assess the safety of slopes through a "factor of safety," defined as the ratio of strength available along a potential failure surface to that required for failure to occur. With the exception of the Swedish method (simple or ordinary method of slices), all methods are based on the assumption that the "overall factor of safety" is the same as the "local factor of safety" at any point along a potential slip surface (5,19,35). Limit equilibrium methods cannot be used to simulate construction history nor to study the influence of initial stresses on the slope's safety.

The versatile finite element method has been used to provide stresses and deformations within soil masses for more than two decades. However, sophisticated formulations of this method are necessary for its application to geological media. It should be noted that these formulations require input data that is neither readily available nor easy to obtain. Also, quantitative interpretation of results necessitates some form of limit equilibrium calculations (10,18). These factors have ensured not only the survival but also the continued popularity of limit equilibrium methods as evidenced by the work of Sarma (24).

The literature is filled with case studies of slopes that have failed, although their calculated factors of safety were greater than one. Consequently, geotechnical engineers have learned to accept the fact that there is always some probability of failure regardless of the numerical value of the safety factor obtained by any conventional formula or method. In recent years, considerable attention has been given to the identification and description of the uncertainties that exist in soil properties, pore water pressures, environmental conditions, and the theoretical formulations of soil behavior. Tools available in the field of reliability analysis have been introduced to assess the probability of failure of slopes and its dependence on such uncertainties (1,2,16,17,20,36,37). Some progress has also been made to evaluate the most probable extent or length of the failure surface (lateral extent of failure) within an embankment (33). Following Vanmarcke's novel approach, the most probable length of failure of a natural slope has also been considered (14). All these studies, however, are concerned with the event of complete (simultaneous) failure and little attention has been given to failure progression within a slope.

Formulation of Model

Modes of Failure

Several modes of failure initiation and progression are possible, e.g., failure may start at one extremity of a slip surface and progress towards the other extremity; it may start somewhere within the slip surface and propagate in either direction, i.e., towards the crest or the toe of the slope. The recognition of a probable failure mode is facilitated by good geological and geotechnical information and requires experience as well as engineering judgment. In excavated slopes, e.g., stress concentrations at the toe may lead to failure propagation upwards from the toe (8,13,21); or, in the case of natural slopes, failure may start at the crest (12,22). For embankments, the likelihood of failure initiating from the crest has been demonstrated by Romani, et al. (23), and centrifuge tests on soil models of built-up slopes appear to confirm this (3).

On the basis of the distributions of effective normal and shear stresses, it has been suggested (6,32) that long-term failures are likely to propagate from the ends of a slip surface (crest or toe) while short-term, undrained failures of clay slopes may develop in the interior first and then propagate outwards.

The present model can, in principle, handle all these possibilities. Only one mode is considered in detail, namely the case where failure initiates at the lower extremity and propagates upwards along a slip surface. The slices are numbered consecutively from 1 to n starting from the bottom of a slope (Fig. 1).

Definition of Local Failure

Direct observations of slope failure and inferences drawn from many case rec-

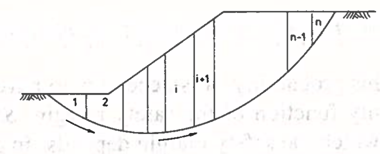


FIG. 1.—Slip Surface within a Slope Potential Sliding Mass Subdivided into a Number of Segments of Slices

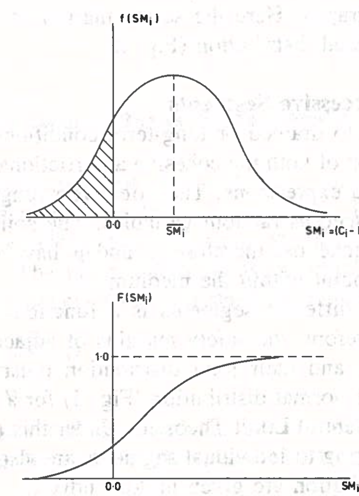


FIG. 2.—Normal Distribution of Random Variable

ords have confirmed the existence of slip surfaces as boundaries of earth masses subject to sliding. Therefore, it is desirable to retain the concept of a potential slip surface while modelling progressive failure.

In developing the present model, it is assumed that the soil mass above the slip surface is subdivided into a number of vertical slices (Fig. 1). The base of each slice is, thus, a segment of the potential slip surface. Failure may start at the base of any slice and then extend successively to other slices. In order to study the probability of failure progressing in this manner, the capacity, C_i, (shearing resistance) and the demand, D_i, (shear force) along each segment i ($i = 1, \ldots, n$) of the slip surface must be evaluated. The safety margin SM_i of the ith segment is given by the difference between its capacity and demand, i.e., $SM_i = (C_i - D_i)$. Failure is defined as the event whereby the safety margin receives values less than or equal to zero, i.e.

$$(\text{Failure})_i = [SM_i \leq 0] = [(C_i - D_i) \leq 0] \quad \text{(1a)}$$

in which $C_i = c\,L_i + N_i\mu;\quad D_i = W_i \sin\alpha_i$ . . . . . (1b)

in which c = the cohesion parameter: $\mu(= \tan\phi)$ the friction parameter; W_i = the weight of the slice; N_i = the effective normal stress acting on the base of the slice; and α_i = the inclination of the base to the horizontal.

The probability of local failure (at the base of the ith slice) is then equal to

$$p_{fi} = \mathbf{P}\,[SM_i \leq 0] \quad \text{(2)}$$

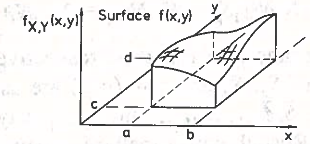
In order to determine this probability, it is necessary to have previous knowledge of the probability density function of the safety margin, SM_i, or of the random variables, c and μ, on which the safety margin depends. In general, the statistical values, e.g., mean values and variances, of the shear strength parameters are evaluated, first, from shear strength data, and then they are substituted in the appropriate expression of the probability of failure. More attention is given to this in subsequent paragraphs. Here the safety margin of any slice is assumed to follow a standard normal distribution (Fig. 2).

Interdependence of Successive Segments

Failure corresponding to drained or long-term conditions is considered here, necessitating the inclusion of both the cohesive and frictional parameters of shear strength in the developed expressions. The effective strength parameters, c and μ (= tan ϕ), are introduced as random variables. The soil material is assumed to be statistically homogeneous, therefore, c and μ have the same probability density function at any point within the medium.

The safety margin of different segments is a function of the same random variables, c and μ. Therefore, the safety margins of adjacent slices are not independent of each other and their joint distribution must be considered. The assumption of a bivariate normal distribution (Fig. 3) for a pair of adjacent segments accords with the Central Limit Theorem. Under this assumption, the marginal distributions belonging to individual segments are also normal. The details of the joint normal distribution are given in Appendix I.

A normal distribution of the safety margin requires only two statistical parameters for its description, viz., the mean value and the variance. The joint normal



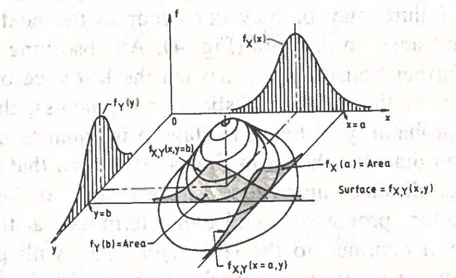


FIG. 3.—Joint Normal Distribution of Two Random Variables

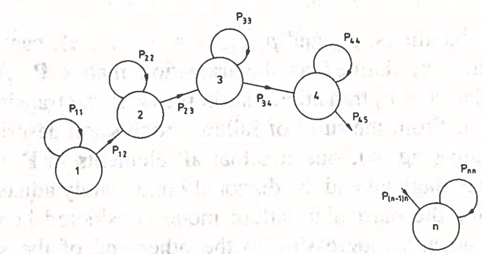


FIG. 4.—Transition Diagram Shows Proposed Model of Failure Progression

distribution of the safety margins, SM_i and SM_{i+1}, of two adjacent slices, i and $i+1$, requires five parameters for its description. These parameters are two mean values, $\overline{SM}_i$ and $\overline{SM}_{i+1}$; two variances, $S^2_{SM_i}$ and $S^2_{SM_{i+1}}$; and the correlation coefficient, $r_{SM_i SM_{i+1}}$. The latter is defined as the ratio of the covariance of SM_i and SM_{i+1}, and the product of the two standard deviations, i.e.

$$r_{SM_i \cdot SM_{i+1}} = \frac{\text{cov}\,(SM_i, SM_{i+1})}{S_{SM_i}\,S_{SM_{i+1}}} \quad \text{(3)}$$

A general expression for the covariance of two functions of the same random variables is given in Appendix I. Using Eq. 19 SM_i and SM_{i+1} are

$$SM_i = c\,L_i + \mu N_i - W_i \sin\alpha_i;\quad SM_{i+1} = c\,L_{i+1} + \mu N_{i+1} - W_{i+1}\sin\alpha_{i+1} \quad \text{(4)}$$

the expression for the covariance cov (SM_i, SM_{i+1}), entering Eq. 3, becomes

$$\text{cov}\,(SM_i, SM_{i+1}) = L_i L_{i+1} S_c^2 + N_i N_{i+1} S_\mu^2 + (L_i N_{i+1} + L_{i+1} N_i)\,\text{cov}\,(c,\mu) \quad \ldots\ldots (5)$$

in which S_c^2 and S_μ^2 are the variances of c and μ, respectively, and cov (c,μ) is their covariance. The values of S_c, S_μ and cov (c,μ) are assumed to be known. Therefore, the values of $\overline{SM}_i$, $\overline{SM}_{i+1}$, $S_{SM_{i+1}}$, and $S_{SM_{i+1}}$ may be calculated. This is done conveniently by using Rosenblueth's method, as summarized in Appendix II. Also cov (SM_i, SM_{i+1}) is calculated from Eq. 5.

Progression of Failure

The state of the progression of failure may be identified by the number of slice along a slip surface to which failure has progressed. Having reached a certain state, transition of failure may or may not occur to the next state, as shown schematically in the transition diagram (Fig. 4). An absorbing state is one from which there is no further transition; e.g., when the last slice of the slip surface has failed (stage $i = n$, the number of slices or segments), there is no further transition, and the probability of being in stage n is equal to one.

From the transition diagram shown in Fig. 4, it is seen that when failure has reached the ith state, all slices up to slice i have failed. At this state, there are two possibilities: Failure progression will either terminate at the ith slice, with probability p_{ii}, or will continue to the $(i + 1)$th state, with probability $p_{i,i+1}$. There is no third option because reversal of failure or healing of the slip surface is considered impossible in an engineering time scale. The present model assumes a continuous progression; i.e., failure cannot jump to a slice across one or more unfailed slices.

The transition probabilities, $p_{i,i}$ and $p_{i,i+1}$, $(i = 1, \ldots, n)$, can be represented in the form of a matrix, defined as the transition matrix **P**. An element $p_{i,j}$ $(i, j = 1, \ldots n)$ of the $[n \times n]$ transition matrix provides the transition probability from state i to state j. From the rules of failure progression described above and the transition diagram (Fig. 4), one has that all elements in **P** are zero except those on the principal diagonal and the diagonal immediately adjacent to its right. (This is true only for the particular failure mode considered here, i.e., failure starting from one end and progressing to the other end of the slip surface. If failure starts somewhere in the middle, there will be more nonzero elements involving the diagonal adjacent to the principal diagonal and on its left.)

Thus, the transition matrix $[P]$ has the following form:

$$[P] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \cdot \\ i \\ i+1 \\ \cdot \\ n \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & \cdot & i & i+1 & \cdot & n \end{array} \\ \left[\begin{array}{cccccccc} p_{1,1} & p_{1,2} & 0 & \cdot & 0 & 0 & \cdot & 0 \\ 0 & p_{2,2} & p_{2,3} & \cdot & 0 & 0 & \cdot & 0 \\ & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & p_{i,i} & p_{i,i+1} & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 & p_{i+1,i+1} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 1 \end{array}\right] \end{array} \quad \ldots (6)$$

The following are two important characteristics of the transition matrix **P**: (1) The sum of the elements of each row must be equal to unity, as they represent probabilities of mutually exclusive and exhaustive events; i.e.

$$p_{i,i} + p_{i,i+1} = 1, \quad i = 1, 2, \ldots, n \quad \ldots\ldots (7)$$

and (2) the last row of the matrix corresponds to the absorbing state and, therefore, $p_{nn} = 1$ and $p_{ni} = 0$, $i = 1, 2 \ldots, n - 1$.

Determination of Transition Probabilities

An element $p_{i,i+1}$, $i = 1, 2 \ldots, n - 1$, in the transitions matrix denotes the probability with which the $(i + 1)$th slice fails given the ith slice has already failed. Thus, $p_{i,i+1}$ is a conditional probability and can be expressed:

$$p_{i,i+1} = \mathbf{P}[(\text{failure})_{i+1} | (\text{failure})_i] \quad \ldots\ldots (8)$$

Introducing the definitions of failure, given by Eq. 1, the aforementioned expressions become

$$p_{i,i+1} = \mathbf{P}\,[SM_{i+1} \le 0 \mid SM_i \le 0] \quad \ldots\ldots (9)$$

The complement $p_{i,i}$ of $p_{i,i+1}$, i.e., the probability with which failure progression terminates at the ith slice, is equal to

$$p_{i,i} = \mathbf{P}\,[SM_{i+1} > 0 \mid SM_i \le 0] \quad \ldots\ldots (10)$$

so that $p_{i,i} + p_{i,i+1} = 1$. Introducing the definition of the conditional probability into the RHS of Eq. 9, $p_{i,i+1}$ may be written:

$$p_{i,i+1} = \frac{\mathbf{P}[(SM_{i\le 0}) \text{ and } (SM_{i+1 \le 0})]}{\mathbf{P}[SM_i \le 0]} \quad \ldots\ldots (11)$$

i.e., the conditional probability is replaced by the ratio of the probability of the two slices, i and $i + 1$, failing simultaneously over the probability of failure of the ith slice.

As SM_i, $i = 1, 2, \ldots, n$, is normally distributed and the joint distribution of SM_i and SM_{i+1} is bivariate normal, one has that the denominator of Eq. 11 is equal to

$$\mathbf{P}[SM_i \le 0] = \int_{-\infty}^{0} f_{SM_i}(x)\,dx = F_{SM_i}(0) \quad \ldots\ldots (12)$$

in which f_{SM_i} and F_{SM_i} are the normal and cumulative normal distributions, respectively. The numerator of Eq. 11 is equal to the cumulative bivariate normal distribution evaluated at zero; i.e.

$$\mathbf{P}[(SM_i \le 0) \text{ and } (SM_{i+1} \le 0)] = \int_{-\infty}^{0}\int_{-\infty}^{0} f_{SM_iSM_{i+1}}(x,y)\,dx\,dy$$

$$= F_{SM_iSM_{i+1}}(0,0) \quad \ldots\ldots (13)$$

in which x and y = dummy variables of integration.

The analytical expression for the bivariate normal distribution $f_{SM_iSM_{i+1}}$ is given in Appendix I and is shown schematically in Fig. 3.

TABLE 1.—Statistical Values of Strength Parameters

Statistical parameter (1)	Strength Parameters: μ (= tan ϕ) (2)	Strength Parameters: c (3)
Mean value	$\bar{\mu} = 0.78$ ($\bar{\phi} = 38°$)	$\bar{c} = 500$ psf (23.94 kN/m^2)
Coefficient of variation	$V_\mu = 10\%$	$V_c = 50\%$

Note: Correlation coefficient $r_{c,\mu} = -0.20$.

The model of progressive failure presented above is illustrated in the following example.

Illustrative Example.—In Fig. 5 an earth slope, having a height $H = 25$ ft (7.62 m) and an angle $\beta = 45°$, is shown. The slope mass is subdivided into nine slices, indicating nine states of progression during failure. The unit weight of the material is equal to $\gamma = 120$ pcf (18.84 kN/m^3), while the statistical values of the strength parameters are given in Table 1.

First, the most conventionally critical failure surface is determined. This is achieved using the computer program STABL (9,26). From among the 10 most

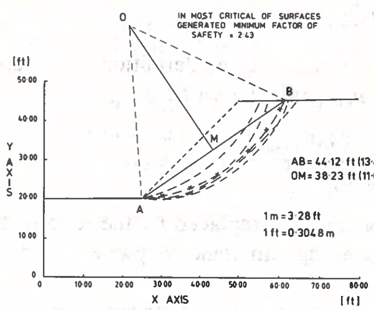


FIG. 5.—Trial Slip Surfaces for Given Problem—Search for Most Critical Surface

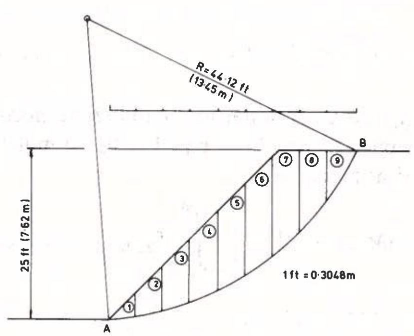


FIG. 6.—Example Problem with Most Critical Slip Surface and Segments Considered for Failure Progression

critical failure surfaces, shown in Fig. 5, curve AB corresponds to the smallest value of the factor of safety FS, which is equal to 2.43.

It is assumed that failure will progress along surface AB (Fig. 6). This is followed by calculations of the shear and normal forces along the base of each slice which enter the expressions for the safety margin, Eq. 1. The mean values and standard deviation of the safety margin, SM_i, of each slice and the covariance and correlation coefficient of the safety margins of successive slices are listed in Table 2. These results were determined using the methods described in Appendices I and II. The numerical values of the probability of failure of each slice are listed in Table 3. These were determined using tables of the standard normal variate. Finally, using Eqs. 11–13, the transition probabilities $p_{i,i+1}$ of progressive failure were determined. The resulting transition probability matrix is

$$[P] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 0.87 & 0.13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.81 & 0.19 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.64 & 0.36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.49 & 0.51 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.31 & 0.69 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.26 & 0.74 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.19 & 0.81 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.65 & 0.35 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \ldots\ldots (14)$$

Analysis

Model—Its Basis and Attributes.—The probabilistic model presented in this paper deals essentially with the progression of failure in space. It is based on the fact that, in practice, a probability of local failure always exists. Various factors that influence failure initiation have already been outlined in the introduction, e.g., stress concentration at the toe of a "safe" slope may initiate failure. Similarly, tension cracks may enhance the tendency for failure initiation at the crest of a slope. Almost three decades back, Bishop (4) noted that local overstress could occur in so-called "safe" embankments. In a typical case, he found that local failure occurred even when the calculated overall safety factor, F_s, was as high as 1.8. Similar results have been noted in numerous stress-deformation studies based on the versatile finite element method (18,35). The actual magnitude of F_s required to prevent local overstress anywhere in a slope depends on several factors, such as the type of slope, the soil properties, the sequence and history of construction, and environmental effects.

Thus, regardless of the calculated local safety margin, SM_1, of the first segment, e.g., Eq. 1, the "real" local safety factor, F_{local}, may be very low because of these factors, and failure may initiate, i.e., $F_{local} << (C_1/D_1)$. It is useful to

TABLE 2.—Statistical Values of Local Safety Margins

Slice number (1)	Mean, $\overline{SM}_i$ (2)	Standard deviation, S_{SM_i} (3)	Covariance, Cov (SM_i, SM_{i+1}) (4)	Correlation coefficient, $r_{SM_i,SM_{i+1}}$ (5)
1	1.10×10^5	4.47×10^4		
			1.87×10^9	0.90
2	1.43×10^5	4.63×10^4		
			1.91×10^9	0.84
3	1.64×10^5	4.91×10^4		
			2.26×10^9	0.81
4	1.79×10^5	5.70×10^4		
			2.61×10^9	0.79
5	1.55×10^5	5.80×10^4		
			2.63×10^9	0.78
6	1.16×10^5	5.83×10^4		
			3.16×10^9	0.80
7	1.00×10^5	6.81×10^4		
			3.98×10^9	0.85
8	0.758×10^5	6.90×10^4		
			5.87×10^9	0.92
9	1.51×10^5	9.22×10^4		

Note: The safety margin of any slice has been expressed as a moment rather than a force; thus, Eq. 4 is multiplied by radius, R, of circular slip surface.

designate the state of no local failure as state 0 (zero), the probability of this state being maintained as p_{00}, and the probability of failure initiation as p_{01}, i.e., transition from no-failure state to the state whereby the first segment has failed. Accordingly, one may write

$$p_{01} = \mathbf{P}\,[F_{\text{local}} \le 0], \quad p_{00} = 1 - p_{01} \quad \text{(15)}$$

Since F_{local} is the actual local safety factor, likely to be much smaller than the calculated ratio of local capacity and local demand, it is obvious that the probability of failure initiation is much greater than that of local failure based on a conventional approach, i.e.

$$F_{\text{local}} << (C_1 - D_1) \therefore p_{01} >> p_{f1} >> \mathbf{P}\,[(C_1 - D_1) \le 0] \quad \text{(16)}$$

Having recognized and accepted that failure can initiate, in so-called "safe" slopes, the need for a suitable basis for assessing the progression of failure becomes absolutely clear. The model presented in this paper enables the extension or propagation of failure to be evaluated logically on a probabilistic basis. Whatever the current state, the probability of continuation in the same state or extension to the next state can be determined. The fact that this is done by considering two adjacent segments at a time acknowledges the interdependence of elements in a soil mass. The assumption that failure can not jump across safe slices appears justified intuitively in light of experience and engineering judgment. It is interesting that these features imply the type of one-step memory used in the well-known Markov chain model. However, the model presented herein should not be confused with a Markov one. In the latter, transitions occur in time, and consecutive states are those occurring at successive time steps.

The proposed model is not dependent on a particular definition of the local safety margin. The Fellenius assumption was used in Eq. 1 for simplicity. Also, as pointed out earlier, other limit equilibrium methods assume local safety factors to be the same as the overall safety factor. In principle, any suitable definition of local safety margin may be used, e.g., one based on the computed stress field or one based on the level of strains or deformation reached at different locations. Almost all previous work on probabilistic study of slope stability is based on, and directly linked with, deterministic models, such as limit equilibrium, and is, therefore, subject to some of their weaknesses. In the approach presented herein, use of such models is optional. Moreover, failure progression can be studied without considering any specific mechanism of strength decrease. It is obvious that strain-softening behavior of soils enhances the tendency for progressive failure. The inclusion of strain-softening behavior in the proposed model is, however, a matter of detail only. Similarly, pore water pressure, a significant variable in slope stability and progressive failure, may be included as an independent random variable without altering the basic simplicity of the formulation.

It may be noted that the assumption of a statistically homogeneous soil mass has been made in this paper. This implies, for instance, that the mean value and variance of each strength parameter are the same everywhere in the soil mass. While, in principle, one could consider the variability of shear strength along the slip surface, in practice it is difficult to secure a sufficient amount of laboratory or field data, or both, necessary to quantify this variability on a statistical basis. Thus, the necessity, the present work was based on the assumption that the probability density function, mean value, and standard deviation of strength were the same anywhere within the medium.

Results.—Table 2 shows a consistently positive and strong correlation between the safety margins of adjacent slices. In this example, the magnitude of the coefficients of correlation (of safety margins of adjacent slices) are within a relatively narrow range, which is to be expected considering the nature of a statistically homogeneous medium. Strong and positive correlation implies physical dependence of the stability of adjacent segments of a soil mass on each other. Therefore, the results are consistent with intuitive judgment. The fact that the prob-

TABLE 3.—Probability of Local Failure

Slice number (1)	Standard normal variate, ν (2)	Probability of failure, $p_{fi} = \mathbf{P}[SM_i \le 0]$ (3)
1	−2.46	6.95×10^{-3}
2	−3.09	1.00×10^{-3}
3	−3.34	4.19×10^{-4}
4	−3.14	8.45×10^{-4}
5	−2.67	3.79×10^{-3}
6	−1.99	2.33×10^{-2}
7	−1.47	7.08×10^{-2}
8	−1.10	1.36×10^{-1}
9	−1.64	4.5×10^{-2}

abilities of failure p_{fi} shown in Table 3 vary significantly over the slices might have led to the conclusion that correlation between slices would vary in the same way. However, Tables 2 and 3 show that this is not so.

Turning now to the transition probability matrix, i.e., Eq. 14, the results are noteworthy. At the beginning, the probability of remaining in the initial state is high in comparison to the probability of progression to the next stage. In successive stages, the probabilities of failure progression increase steadily until the penultimate stage. The fact that there is again drop in the magnitude of the probability in this one stage may be a reflection of the geometry of the particular slip surface. One can draw the general conclusion that the probability of failure progression will be influenced by the physical nature of the problem within a given soil medium.

Summary and Conclusions

The primary objective of the present study were to (1) Demonstrate that probabilistic analysis of progressive failure is feasible; and (2) to develop an appropriate model for such an analysis with accompanying formulation. Failure progression was defined suitably as a spatial and continuous extension of the failure zone along a potential slip surface. A statistically homogeneous medium was considered, bearing in mind that sufficient data is usually not available for the determination of variations of statistical parameters along a slip surface. Moreover, only two random variables were introduced in the formulation at this stage, although the inclusion of other random variables is quite feasible. A normal distribution of the safety margin of each slice was considered appropriate. Moreover, in accordance with the Central Limit Theorem, the joint normal distribution of two adjacent slices was assumed to be bivariate normal. A suitable procedure was outlined for determining the five statistical parameters required for the description of this bivariate normal distribution. Expressions for the probability of failure progression (transition probabilities) were derived after specifying the rules of the model, namely: (1) One-step memory, i.e., interdependence of adjacent slices; (2) failure can not jump over unfailed slices; and (3) no healing of the slip surface can occur in an engineering time scale. The model and its formulation were then illustrated by a worked numerical example. The attributes and significance of the model and the significance of the results were then examined.

The following conclusions can be drawn from the present study:

1. A probabilistic formulation of progressive failure is feasible.
2. The proposed model gives insight into the interdependence of the stability of adjacent elements in a soil mass. Consequently, it is potentially valuable in clarifying the real behavior of soil masses, especially slopes.
3. In practice, there is always a probability of local failure. The proposed model provides a powerful tool for studying the extension or *propagation* of such failure on a probabilistic basis.
4. The model can be of practical benefit in most situations, except, if it can be demonstrated in a given case that the actual probability of local failure occurring anywhere in a slope is very low indeed.

5. In a given example, it was shown that the probability of failure progression increases consistently in eight stages or states from 0.13–0.81. The coefficients of correlation of adjacent slices were all high and within a relatively narrow range.
6. The results of any study based on the proposed model must be considered in light of the physical nature of the problem and with an awareness of significant factors influencing stability.
7. The model is not dependent on conventional deterministic concept of a safety margin, and the use of any such concepts is optional. Extensions of the model along significant directions are feasible.

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Appendix I

Covariance of Safety Margins of Adjacent Segments

In general, if there are two functions, R and S, of different variables (x_1, x_2, $x_3 \ldots x_n$) e.g.

$$R = R(x_1, x_2, x_3 \ldots x_n); \quad S = S(x_1, x_2, x_3 \ldots x_n) \quad \ldots \quad (17)$$

$$\text{then} \quad \mathrm{Cov}\,(RS) = \sum_{i=1}^{n} \left(\frac{\partial R}{\partial x_i}\right)\left(\frac{\partial S}{\partial x_i}\right) V[x_i]$$

$$+ \sum_{i<j} \left[\left(\frac{\partial R}{\partial x_i}\right)\left(\frac{\partial S}{\partial x_j}\right) + \left(\frac{\partial R}{\partial x_j}\right)\left(\frac{\partial S}{\partial x_i}\right)\right] \mathrm{Cov}\,(x_i, x_j) \quad \ldots \quad (18)$$

From Eq. 4 in the text and Eq. 18

$$\mathrm{Cov}\,(SM_i, SM_{i+1}) = \left(\frac{\partial SM_i}{\partial c}\right)\left(\frac{\partial SM_{i+1}}{\partial c}\right) S_c^2 + \left(\frac{\partial SM_i}{\partial \mu}\right)\left(\frac{\partial SM_{i+1}}{\partial \mu}\right) S_\mu^2$$

$$+ \left[\left(\frac{\partial SM_i}{\partial c}\right)\left(\frac{\partial SM_{i+1}}{\partial \mu}\right) + \left(\frac{\partial SM_i}{\partial \mu}\right)\left(\frac{\partial SM_{i+1}}{\partial c}\right)\right] \mathrm{Cov}\,(c,\mu) \quad \ldots \quad (19)$$

It is found that

$$\frac{\partial SM_i}{\partial c} = L_i, \quad \frac{\partial SM_{i+1}}{\partial c} = L_{i+1}, \quad \frac{\partial SM_i}{\partial \mu} = N_i, \quad \frac{\partial SM_{i+1}}{\partial \mu} = N_{i+1} \quad \ldots \quad (20)$$

From Eqs. 19 and 20, the final expression shown as Eq. 5 in the text is obtained.

Bivariate Normal Distribution

The bivariate normal probability density function for random variables SM_i, SM_{i+1} may be written:

$$f_{SM_i,SM_{i+1}}(SM_i, SM_{i+1}) = \frac{1}{2\pi S_{SM_i} S_{SM_{i+1}} \sqrt{1-r^2}}$$

$$\exp\left\{\frac{1}{-2(1-r^2)}\left[\left(\frac{SM_i - \overline{SM}_i}{S_{SM_i}}\right)^2 - 2r\left(\frac{SM_i - \overline{SM}_i}{S_{SM_i}}\right)\left(\frac{SM_{i+1} - \overline{SM}_{i+1}}{S_{SM_{i+1}}}\right)\right.\right.$$

$$\left.\left. + \left(\frac{SM_{i+1} - \overline{SM}_{i+1}}{S_{SM_{i+1}}}\right)^2\right]\right\} \quad (21)$$

in which r denotes $r_{SM_i}\, SM_{i+1}$, the correlation coefficient, and

$$-\propto < SM_i < \propto; \quad -\propto < SM_{i+1} < \propto; \quad S_{SM_i} > 0; \quad S_{SM_{i+1}} > 0; \quad -1 \le r \le 1;$$

$$-\propto < \overline{SM}_i < \propto; \quad -\propto < \overline{SM}_{i+1} < \propto \quad (22)$$

and $\mathbf{P}\,[x_1 \le SM_i \le x_2$ and $y_1 \le SM_{i+1} \le y_2]$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(SM_i, SM_{i+1}, \overline{SM}_i, \overline{SM}_{i+1}, S_{SM_i}, S_{SM_{i+1}}, r)\, d\,SM_i\, d\,SM_{i+1} \quad (23)$$

Appendix II

Rosenblueth's Method

The safety margin for the ith slice is a function of two random variables

$$SM_i = SM_i\,(c, \mu) \quad (24)$$

To use Rosenblueth's method, the Eqs. 25–27 are necessary:

$$\mathbf{P}_{++} = \mathbf{P}_{--} = \frac{1+r}{4}, \quad \mathbf{P}_{+-} = \mathbf{P}_{-+} = \frac{1-r}{4} \quad (25)$$

$$SM_{i++} = SM_i\,(\bar{c} + S_c, \bar{\mu} + S_\mu); \quad SM_{i+-} = SM_i\,(\bar{c} + S_c, \bar{\mu} - S_\mu);$$

$$SM_{i-+} = SM_i\,(\bar{c} - S_c, \bar{\mu} + S_\mu); \quad SM_{i--} = SM_i\,(\bar{c} - S_c, \bar{\mu} - S_\mu) \quad (26)$$

$$E\,[SM_i^n] = \mathbf{P}_{++}(SM_{i++})^n + \mathbf{P}_{+-}\,(SM_{i+-})^n$$

$$+ \mathbf{P}_{-+}\,(SM_{i-+})^n + \mathbf{P}_{--}(SM_{i--})^n \quad (27)$$

By definition $\overline{SM}_i = E\,[SM_i]$ (28)

$$\text{Var}\,(SM_i) = E\,[SM_i^2] - (\overline{SM}_i)^2 \quad (29)$$

$$S_{SM_i} = [\text{Var}\,(SM_i)]^{1/2} \quad (30)$$

Values of $\bar{c}$, $\bar{\mu}$, S_c, and S_μ that appear in Eq. 26 are given and are common for

all SM_i values. The correlation coefficient r (i.e., $r_{c\mu}$) is also known. A simple program was developed to use Rosenblueth's method on a hand calculator.

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Appendix IV.—Notation

The following symbols are used in this paper:

C_i = capacity of slice i;
c = cohesion;
$\bar{c}, S_c$ = mean of cohesion and its standard deviation;

cov = covariance;
D_i = demand of slice i;
F_{local} = local factor of safety;
$\mathbb{P}(\ )$ = probability of an event;
$p_{i,i+1}$ = probability of failure progression from state i to state $(i + 1)$;
R = radius of slip surface;
r = correlation coefficient;
SM_i = safety margin of slice i;
$\overline{SM}, S_{SM}$ = mean of safety margin and its standard deviation;
V = coefficient of variation;
W_i = weight of slice i;
α_i = inclination of the base of slice i to the horizontal;
μ = tan ϕ = friction parameter; and
ϕ = angle of internal friction.