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DISCUSSION

Proc. Paper 7785

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[a] Discussion period closed for this paper. Any other discussion received during this discussion period will be published in subsequent Journals.

APPENDIX II.—NOTATION

The following symbols are used in this paper:

B = width of foundation;
C_D = dilational wave velocity;
C_S = shear wave velocity;
E = Young's modulus of soil;
E_b = Young's modulus of beam;
G = shear modulus;
I = moment of inertia for beam;
K = coefficient;
k_s = subgrade modulus;
$(k_s)_1$ = subgrade modulus for foundation 1 ft wide;
k_v = vertical spring constant;
k_ϕ = rotational spring constant;
L = length of straight beam or of segment of beam;
x = distance along beam from concentrated load;
y = deflection of beam;
y_o = deflection of beam beneath concentrated load;
W = concentrated load acting on beam;
λ = parameter in theory for beam on elastic foundation; and
ν = Poisson's ratio.

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LIMIT ANALYSIS OF STABILITY OF SLOPES

By Wilfred F. Chen,[1] A. M. ASCE and M. W. Giger[2]

INTRODUCTION

The upper bound theorem of the generalized theory of perfect plasticity has been previously applied to obtain the critical height of an embankment (1). A rotational failure mechanism (logarithmic spiral) passing through the toe was assumed in the analysis (1). These limit analysis results were found to be in good agreement with existing limit equilibrium solutions. The case where the failure plane may pass below the toe, as for small values of friction angle ϕ and embankment slope angle β (Fig. 1), was not considered. This will be described herein. The soil is assumed, as in Ref. 1, to be a perfectly plastic material which obeys the Coulomb yield criterion and its associated flow rule (2).

CRITICAL HEIGHT OF EMBANKMENT

The upper bound theorem of limit analysis states that the embankment shown in Fig. 1 will collapse under its own weight if, for any assumed failure mechanism, the rate of external work done by the soil exceeds the rate of internal energy dissipation. Equating external and internal energies for any such mechanism thus gives an upper bound on the critical height.

The rate of external work done by region ABC'CA can easily be obtained by first finding rates of work $\dot{W}_1$, $\dot{W}_2$, $\dot{W}_3$, and $\dot{W}_4$ due to the soil weight in regions OBC'O, OABO, OAC'O, and ACC'A, respectively. The rate of external work for region ABC'CA is then obtained by the simple algebraic summation, $\dot{W}_1 - \dot{W}_2 - \dot{W}_3 - \dot{W}_4$. It is found, after performing some algebraic manipulations, that the total rate of external work due to the weight of the soil in re-

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gion ABC'CA is

$$\gamma\,\Omega\, r_o^3\,(f_1 - f_2 - f_3 - f_4) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

in which γ = the unit weight of the soil and Ω = the angular velocity of region ABC'CA. Functions f_1, f_2, and f_3 remain identical in their form, as in the previous solution (1), and function f_4 resulting from region ACC' is expressed as

$$f_4 = \left(\frac{H}{r_o}\right)^2 \frac{\sin(\beta - \beta')}{2\sin\beta\sin\beta'}\left[\cos\theta_o - \left(\frac{L}{r_o}\right)\cos\alpha - \frac{1}{3}\left(\frac{H}{r_o}\right)(\cot\beta' + \cot\beta)\right] \quad \ldots\ldots\ldots \quad (2)$$

in which θ_o, θ_h, and β' = angular variables, specifying the assumed mechanism completely; height = H; length of OB = r_o; length of AB = L; and length of C'C = d (Fig. 1).

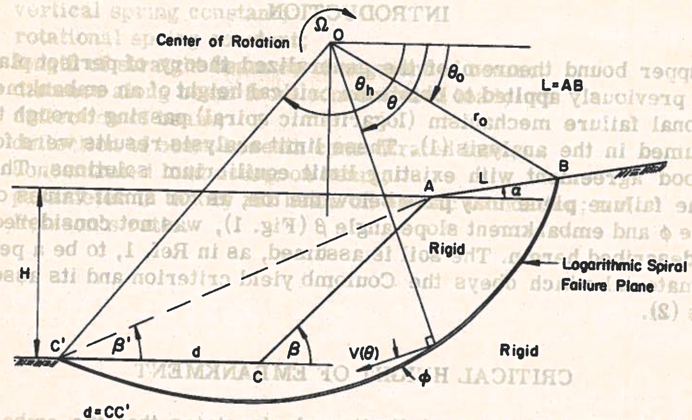

FIG. 1.—FAILURE MECHANISM FOR STABILITY OF SLOPES WITH FAILURE PLANE PASSING BELOW TOE

The internal dissipation of energy occurs along discontinuity surface BC'. The total internal dissipation of energy is identical to the expression given in Ref. 1.

Equating external rate of work, Eq. 1, with internal rate of energy dissipation yields

$$H = \frac{c}{\gamma} f(\theta_h, \theta_o, \beta') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

in which $f(\theta_h, \theta_o, \beta')$ is now defined as

$$f(\theta_h, \theta_o, \beta') = \frac{\sin\beta'\,\{\exp[2(\theta_h - \theta_o)\tan\phi] - 1\}}{2\sin(\beta' - \alpha)\tan\phi\,(f_1 - f_2 - f_3 - f_4)}\{\sin(\theta_h + \alpha)\exp[(\theta_h - \theta_o)\tan\phi] - \sin(\theta_o + \alpha)\} \quad \ldots\ldots \quad (4)$$

TABLE 1.—STABILITY FACTOR $N_s = H_c(\gamma/c)$ BY LIMIT ANALYSIS FOR SMALL VALUES OF α, β, and ϕ

Friction angle, ϕ, in degrees (1)	Slope angle, α, in degrees (2)	Slope angle, β, in degrees							
		50 (3)	45 (4)	40 (5)	35 (6)	30 (7)	25 (8)	20 (9)	15 (10)
0	0	5.52[a]	5.53[a]	5.53[a]	5.53[a]	5.53	5.53	5.53	5.53
5	0	6.92[a]	7.35[a]	7.84[a]	8.41[a]	9.13	10.02	11.46	14.38
5	5	6.76[a]	7.18[a]	7.64[a]	8.19[a]	8.83	9.65	10.99	13.71

[a] Failure through toe; all others failure below toe.

TABLE 2.—COMPARISON OF STABILITY FACTOR $N_s = H_c(\gamma/c)$ BY METHODS OF LIMIT EQUILIBRIUM AND LIMIT ANALYSES $\alpha = 0$

Slope angle, β, in degrees (1)	Friction angle, ϕ, in degrees (2)	CURVED FAILURE SURFACE			
		Limit Equilibrium (3)			Limit Analysis
		Slices (3)	ϕ Circle (4)	Log-Spiral (5)	Log-Spiral (6)
90	0	3.83	3.83	3.83	3.83
	5	4.19	4.19	4.19	4.19
	15	5.02	5.02		5.02
	25	6.06	6.06	6.06	6.06
75	0	4.57	4.57	4.57	4.56
	5	5.13	5.13		5.14
	15	6.49	6.52		6.57
	25	8.48	8.54		8.58
60	0	5.24	5.24	5.24	5.25
	5	6.06	6.18	6.18	6.16
	15	8.33	8.63	8.63	8.63
	25	12.20	12.65	12.82	12.74
45	0	5.88	5.88[a]	5.88[a]	5.53[a]
	5	7.09	7.36		7.35
	15	11.77	12.04		12.05
	25	20.83	22.73		22.90
30	0	6.41[a]	6.41[a]	6.41[a]	5.53[a]
	5	8.77[a]	9.09[a]		9.13[a]
	15	20.84	21.74		21.69
	25	83.34	111.1	125.0	119.93
15	0	6.90[a]	6.90[a]	6.90[a]	5.53[a]
	5	13.89[a]	14.71[a]	14.71[a]	14.38[a]
	10		43.62[a]		45.49

[a] Critical failure surface passes below toe.

TABLE 3.—STABILITY FACTOR $N_s = H_c(\gamma/c)$ BY LIMIT ANALYSIS

Friction angle, ϕ, in degrees (1)	Slope angle, α, in degrees (2)	Slope angle, β, in degrees															
		90 (3)	85 (4)	80 (5)	75 (6)	70 (7)	65 (8)	60 (9)	55 (10)	50 (11)	45 (12)	40 (13)	35 (14)	30 (15)	25 (16)	20 (17)	15 (18)
0	0	3.83	4.08	4.33	4.56	4.80	5.03	5.25	5.46	5.52	5.53	5.53	5.53	5.53	5.53	5.53	5.53
5	0	4.19	4.50	4.82	5.14	5.47	5.81	6.16	6.53	6.92	7.35	7.84	8.41	9.13	10.02	11.46	14.38
	5	4.14	4.44	4.74	5.05	5.37	5.69	6.03	6.38	6.76	7.18	7.64	8.19	8.83	9.65	10.99	13.71
10	0	4.58	4.97	5.37	5.80	6.25	6.73	7.26	7.84	8.51	9.31	10.30	11.61	13.50	16.64	23.14	45.49
	5	4.53	4.91	5.30	5.71	6.15	6.63	7.14	7.72	8.38	9.16	10.13	11.42	13.28	16.37	22.79	44.95
	10	4.47	4.83	5.21	5.61	6.03	6.48	6.99	7.54	8.18	8.93	9.87	11.11	12.89	15.84	21.96	42.90
15	0	5.02	5.50	6.01	6.57	7.18	7.85	8.63	9.54	10.64	12.05	13.97	16.83	21.69	32.11	69.40	
	5	4.97	5.44	5.94	6.49	7.08	7.75	8.52	9.42	10.51	11.91	13.82	16.65	21.48	31.85	69.05	
	10	4.91	5.36	5.85	6.38	6.97	7.63	8.38	9.26	10.34	11.72	13.59	16.38	21.14	31.38	68.26	
	15	4.83	5.27	5.74	6.26	6.82	7.46	8.19	9.04	10.09	11.42	13.23	15.92	20.49	30.25	65.17	
20	0	5.50	6.10	6.75	7.48	8.30	9.25	10.39	11.80	13.63	16.16	19.99	26.66	41.22	94.63		
	5	5.46	6.04	6.68	7.40	8.21	9.16	10.28	11.69	13.51	16.03	19.85	26.48	41.02	94.38		
	10	5.40	5.97	6.60	7.30	8.10	9.04	10.16	11.54	13.35	15.85	19.64	26.23	40.69	93.78		
	15	5.33	5.88	6.50	7.18	7.97	8.89	9.98	11.35	13.12	15.58	19.32	25.82	40.09	92.90		
	20	5.24	5.77	6.37	7.03	7.79	8.68	9.74	11.07	12.79	15.17	18.77	25.01	38.64	88.63		
25	0	6.06	6.79	7.62	8.58	9.70	11.05	12.74	14.97	18.10	22.90	31.33	50.06	119.93			
	5	6.01	6.73	7.56	8.50	9.61	10.96	12.64	14.86	17.98	22.77	31.19	49.89	119.70			
	10	5.95	6.67	7.48	8.41	9.51	10.84	12.52	14.73	17.83	22.60	30.99	49.63	119.35			
	15	5.89	6.58	7.38	8.30	9.38	10.70	12.36	14.55	17.62	22.35	30.69	49.23	118.79			
	20	5.80	6.48	7.26	8.16	9.22	10.51	12.14	14.30	17.33	21.98	30.20	48.50	117.43			
	25	5.70	6.35	7.10	7.97	9.00	10.26	11.84	13.92	16.85	21.35	29.24	46.76	112.07			
30	0	6.69	7.61	8.67	9.94	11.48	13.44	16.04	19.71	25.41	35.54	58.27	144.20				
	5	6.64	7.55	8.61	9.86	11.40	13.35	15.94	19.61	25.29	35.41	58.13	144.01				
	10	6.59	7.48	8.53	9.77	11.30	13.24	15.82	19.48	25.15	35.25	57.92	143.74				
	15	6.52	7.40	8.44	9.67	11.18	13.10	15.67	19.31	24.96	35.01	57.63	143.31				
	20	6.44	7.31	8.32	9.54	11.03	12.93	15.47	19.08	24.68	34.67	57.16	142.54				
	25	6.35	7.19	8.18	9.37	10.83	12.70	15.20	18.74	24.27	34.11	56.30	140.54				
	30	6.22	7.04	7.99	9.14	10.56	12.37	14.78	18.22	23.54	33.01	54.25	134.52				
35	0	7.42	8.58	9.97	11.68	13.86	16.77	20.94	27.45	39.11	65.52	166.38					
	5	7.38	8.52	9.90	11.60	13.77	16.68	20.84	27.34	39.00	65.39	166.22					
	10	7.32	8.46	9.82	11.51	13.68	16.58	20.73	27.22	38.85	65.22	166.00					
	15	7.26	8.38	9.73	11.41	13.56	16.44	20.58	27.05	38.66	64.70	165.72					
	20	7.19	8.29	9.63	11.29	13.42	16.29	20.40	26.84	38.40	64.65	165.19					
	25	7.10	8.18	9.49	11.13	13.23	16.07	20.14	26.53	38.02	64.12	164.30					
	30	6.99	8.04	9.33	10.93	12.99	15.78	19.78	26.07	37.38	63.14	162.33					
	35	6.84	7.86	9.10	10.64	12.64	15.34	19.21	25.27	36.15	60.80	154.98					
40	0	8.29	9.77	11.61	13.97	17.15	21.72	28.91	41.89	71.49	185.49						
	5	8.24	9.71	11.54	13.89	17.07	21.63	28.82	41.78	71.37	185.35						
	10	8.19	9.65	11.46	13.81	16.97	21.53	28.71	41.66	71.23	185.17						
	15	8.13	9.57	11.38	13.71	16.86	21.40	28.57	41.51	71.04	184.93						
	20	8.06	9.49	11.27	13.59	16.72	21.25	28.39	41.29	70.78	184.57						
	25	7.98	9.38	11.15	13.44	16.55	21.05	28.15	41.00	70.41	184.04						
	30	7.87	9.25	10.99	13.25	16.33	20.78	27.82	40.58	69.81	183.01						
	35	7.74	9.09	10.78	13.00	16.02	20.39	27.32	39.88	68.73	180.81						
	40	7.56	8.86	10.50	12.64	15.55	19.77	26.45	38.53	66.12	172.51						

function $f(\theta_h, \theta_o, \beta')$ has a minimum and, thus, indicates a least upper bound when θ_h, θ_o, and β' satisfy conditions

$$\frac{\partial f}{\partial \theta_h} = 0; \quad \frac{\partial f}{\partial \theta_o} = 0; \quad \frac{\partial f}{\partial \beta'} = 0 \quad \ldots\ldots (5)$$

with $\beta' \leq \beta$ (Fig. 1). The corresponding values for θ_h, θ_o, and β' satisfying

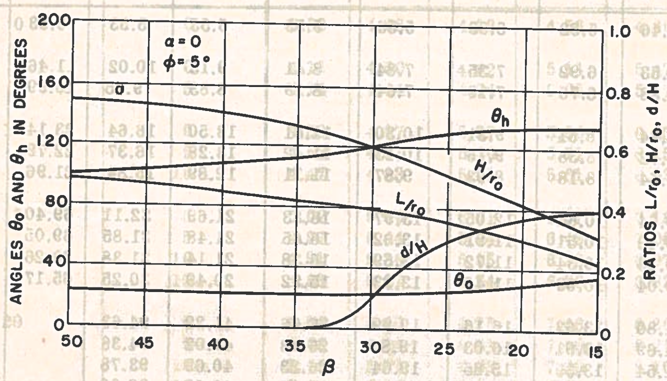

FIG. 2.—CRITICAL VALUES FOR θ_h, θ_o, L/r_o, H/r_o, AND d/H AS FUNCTION OF SLOPE ANGLE β

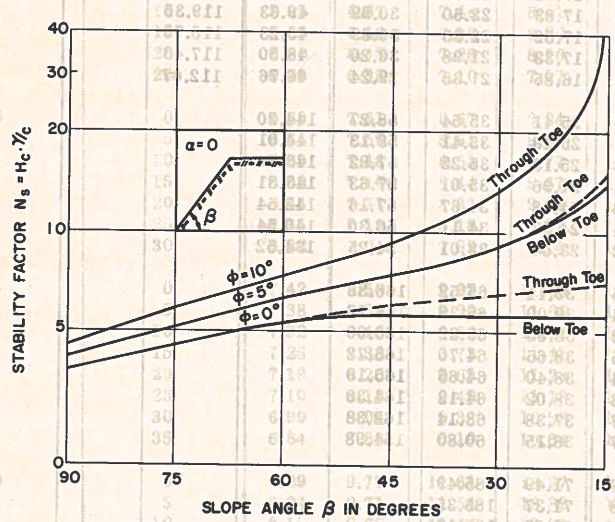

FIG. 3.—STABILITY FACTORS N_s AS FUNCTION OF β

Eq. 5 result in $N_s = \min f(\theta_h, \theta_o, \beta')$. Thus, the critical height becomes

$$H_c \leq \frac{c}{\gamma} N_s \quad \ldots\ldots (6)$$

and $$\frac{d}{H_c} = \frac{\sin(\beta - \beta')}{\sin\beta \sin\beta'} \quad \ldots\ldots (7)$$

which is the ratio between distance d and critical height H_c (Fig. 1).

NUMERICAL RESULTS

A complete numerical solution to this problem has been obtained by numerical methods, the numerical work being performed on a CDC 6400 digital computer.

The results are tabulated numerically in Table 1 for small values of α, β, and ϕ. For the case of $\phi = 5°$ and $\alpha = 0°$ the corresponding critical values of θ_h, θ_o, H/r_o, L/r_o, and d/H are plotted in Fig. 2. Fig. 3 shows the transition zone where the most critical failure plane starts to pass below the toe, when $\alpha = 0$. Comparison of limit analysis results with already existing limit equilibrium solutions are given in Table 2. It may be seen that agreement is good. Table 3 tabulates the value of N_s for various combination of slopes α and β. Because there are no existing solutions available for the case $\alpha \neq 0$, the tabulated results will be useful in the analysis and design of such problems.

CONCLUSIONS

The agreement between the limit analysis and limit equilibrium results relative to the stability of slopes proves both interesting and useful. It can be concluded, therefore, that the upper bound theory of limit analysis may be applied to predict the critical height of slopes. In many cases it may be much more convenient to use the upper bound method and it also places the matter of stability analysis on a much more logical ground.

ACKNOWLEDGMENTS

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The writers thank Mrs. Jane Lenner for typing the manuscript and John Gera for the figure drafting.

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APPENDIX II.—NOTATION

The following symbols are used in this paper:

c = cohesion;
f_1 = function defined in Eq. 4, see Ref. 1;
f_2 = function defined in Eq. 5, see Ref. 1;
f_3 = function defined in Eq. 6, see Ref. 1;
f_4 = function defined in Eq. 2;
f = function defined in Eq. 4;
H, H_c = height and critical height of embankment, respectively;
L, d = lengths (Fig. 1);
N_s = stability factor;
$r_o, r(\theta)$ = length variables of logarithmic spiral curve;
$V(\theta)$ = discontinuous velocity across failure plans (Fig. 1);
α, β = slope angles;
γ = unit weight;
$\theta_h, \theta_o, \beta'$ = angular variables (Fig. 1);
ϕ = friction angle of soil; and
Ω = angular velocity.

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RESPONSE OF NONUNIFORM SOIL DEPOSITS TO TRAVELLING SEISMIC WAVES

By Houshang Dezfulian,[1] A. M. ASCE and H. Bolton Seed,[2] M. ASCE

INTRODUCTION

Records of ground motions in recent earthquakes have provided convincing evidence of the important effects of local soil conditions on the amplitude and frequency characteristics of ground surface motions. Marked variations in surface motions may occur over relatively short distances due to variations either in depth or character of the underlying soil deposits. Such variations are likely to occur in areas adjacent to sloping rock surfaces such as that shown in Fig. 1, and methods of evaluating these variations by determining the response of the soil deposit to motions developed in the underlying rock have been presented. In these analyses, however, the rock motions have been treated as only a time-dependent phenomenon with the same motions being developed at all points in the rock at any given instant. Possible spacial variations in rock motions, due to wave propagation effects, have not been considered.

While the present state of knowledge with regard to spacial variations in seismic motions is quite deficient, the possibility of some type of travelling wave effect outside the epicentral region might well be anticipated. Such an effect might also be particularly important at sites adjacent to a sloping rock surface (as in Fig. 1) where the motions might propagate towards or away from the rock outcrop.

A seismic wave travelling through a number of geologic formations before it reaches an engineering work would undergo a multitude of reflections, re-

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