VOL.97 NO.SM2. FEB. 1971

JOURNAL OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION

PROCEEDINGS OF
THE AMERICAN SOCIETY
OF CIVIL ENGINEERS

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

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This Journal is published monthly by the American Society of Civil Engineers. Publications office is at 345 East 47th Street, New York, N.Y. 10017. Address all ASCE correspondence to the Editorial and General Offices at 345 East 47th Street, New York, N.Y. 10017. Allow six weeks for change of address to become effective. Subscription price is $16.00 per year with discounts to members and libraries. Second-class postage paid at New York, N.Y. and at additional mailing offices. SM.

[a] Discussion period closed for this paper. Any other discussion received during this discussion period will be published in subsequent Journals.

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APPENDIX IV.—NOTATION

The following symbols are used in this paper:

C = phase velocity at the surface;
C_s = shear wave velocity;
f = driving frequency;
G = shear modulus;
H = layer thickness;
l = wavelength of shear waves;
n = mode of propagation;
γ = unit weight;
λ = observed surface wavelength; and
ω = angular frequency.

Journal of the
SOIL MECHANICS AND FOUNDATIONS DIVISION
Proceedings of the American Society of Civil Engineers

UNCONFINED TRANSIENT SEEPAGE IN SLOPING BANKS

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INTRODUCTION

Prior to revetment construction along the Mississippi River, the riverbank slopes should be graded to help ensure their stability under various river stages including drawdown. For a fall in the river level, the free-water or the phreatic surface in the earth bank lags behind the falling level of water in the river, and it is generally difficult to compute such a free-water surface. Conventionally, the designs were based on the free-water surface that results after full drawdown. This procedure is, however, conservative for many cases and requires slopes flatter than necessary (9).

The stability of an earth slope subjected to the effects of changing river stages is dependent on, among other factors, the pore pressures induced within the earth mass due to the seepage. The pore pressures are generally estimated from flow net analysis obtained under steady state conditions. However, a more precise determination of pore pressures is warranted for the case of a continuously moving free surface. In recent years, extensive piezometer installations at selected locations along the banks of the Mississippi River have provided valuable data on the development of free surface and pore pressures as functions of changing river levels.

As a result, the Lower Mississippi Valley Division (LMVD) of the Corps of Engineers asked the Waterways Experiment Station (WES) to investigate the transient flow in earth banks under conditions of variable rates of rise or fall in river level, and to evolve some rational methods for predicting the location of free surface and distribution of pore pressures for use in design and sta-

Note.—Discussion open until July 1, 1971. To extend the closing date one month, a written request must be filed with the Executive Director, ASCE. This paper is part of the copyrighted Journal of the Soil Mechanics and Foundations Division, Proceedings of the American Society of Civil Engineers, Vol. 97, No. SM2, February, 1971. Manuscript was submitted for review for possible publication on May 28, 1970.

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bility analysis. The study described herein forms a part of these investigations.

Closed form solutions to the governing equation of the transient flow are available for simple material and boundary conditions (4,7,10,15,23,27). Increased use of high speed computers have now made possible the use of certain numerical techniques that allow the introduction of complex material and boundary conditions. Numerical methods for the equivalent problems in unsteady gas and heat flow equations have been developed (3,5,6,12,14,19,22,25, 29,30,32).

The problem of unconfined seepage through porous soils is, however, complicated due to the occurrence of the so called phreatic surface and the surface of seepage, and requires special schemes for handling them. In a previous study, Schnitter and Zeller (28) considered flow in earth dams, and on the basis of a simplified falling surface, employed a numerical procedure for the closed form solution of the Boussinesq equation. A parallel plate viscous flow model was used for comparisons with the analytical solutions. On the basis of an approximate approach (8), Newlin and Rossier (21) obtained comparisons between their analytical results and experiments with a polyethylene bead model. In most of the previous work, the applications were restricted to homogeneous, isotropic materials, relatively short model lengths, and a well defined impervious surface (core). A finite difference scheme called the alternating direction explicit procedure (ADEP) (2,20,26), found to be computationally stable and efficient for arbitrary magnitude of timewise subdivision, is employed here. Consideration is given to material nonhomogenity and anisotropy, and because in a long river bank, a well defined impervious core boundary usually is not present, a special scheme is used to define a fictitious boundary that changes with time. An approximate nonlinear form of the equation of free surface flow and its linearized version are used, and the free surface is located by using an interpolation procedure, and by approximately satisfying the boundary condition of zero pressure at that surface. The surface of seepage is determined on the basis of Pavlosky's method of fragments (13,16,18). To ascertain the validity of the numerical procedure, experimental verification is obtained by using a large parallel plate viscous flow model (11,17,31). Satisfactory agreements are obtained between the numerical and experimental results for isotropic homogeneous cases. It is observed that the results from the nonlinear equation show better correlation with the experiments than those from the linearized version. A typical flow net is constructed from numerical results to indicate the applicability of the method to field situations.

GOVERNING EQUATIONS

The following nonlinear equation is adopted in this study (16,23):

$$n\,\frac{\partial h}{\partial t} = \frac{k_x}{2}\,\frac{\partial^2 h^2}{\partial x^2} + \frac{k_y}{2}\,\frac{\partial^2 h^2}{\partial y^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (1a)$$

in which h = head at point (x,y) at time t, $h(x,y,t)$; n = the porosity; k_x and k_y = the horizontal and the vertical permeabilities, respectively; x,y = the space coordinates; and t = the time coordinate. Eq. 1a is based on a number of assumptions such as laminar flow, incompressible fluid, rigid soil, and no change in k_x, k_y, and n with time. Moreover, Dupuit's

assumption is made in deriving Eq. 1a, and the function h^2 satisfies only approximately the basic Laplace equation governing the flow (16).

If a mean head, $\bar{h}$, is assumed, a linearized equation can be obtained (4,11, 13):

$$n\,\frac{\partial h}{\partial t} = \bar{h}\left(k_x\,\frac{\partial^2 h}{\partial x^2} + k_y\,\frac{\partial^2 h}{\partial y^2}\right) \quad \ldots\ldots\ldots\ldots\ldots \quad (1b)$$

The boundary conditions associated with Eq. 1 are, [see Fig. 1(a)]: (1) $h(x,y, 0) = 0$ for an initially dry bank; (2) $h(x, y, t) = f(t)$ on the entrance face; (3)

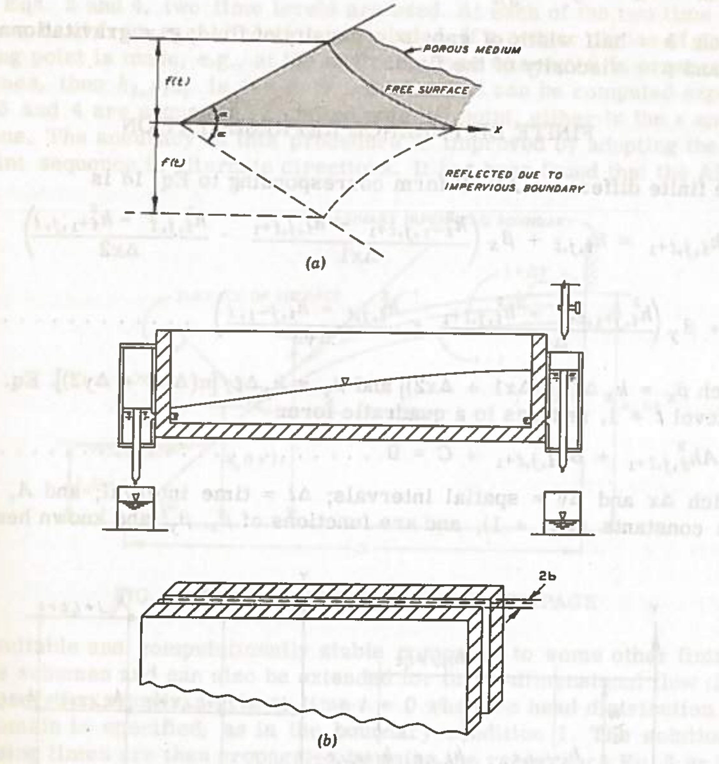


FIG. 1.—(a) SECTION OF PERVIOUS EARTH BANK WITH IMPERVIOUS BASE; (b) PARALLEL PLATE MODEL

$\partial h/\partial y = 0$ at the impervious base; and (4) $h(x, y, t)$ = elevation head at the free surface.

One-dimensional equations corresponding to Eqs. 1a and 1b are

$$n\,\frac{\partial h}{\partial t} = \frac{k_x}{2}\,\frac{\partial^2 h^2}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (2a)$$

and $$n\,\frac{\partial h}{\partial t} = k_x\bar{h}\,\frac{\partial^2 h}{\partial x^2} \quad \ldots\ldots\ldots\ldots\ldots \quad (2b)$$

The previous equations of free surface flow through porous media are considered analogous to the equations governing the flow through the narrow gap of a parallel plate model (17,31). The parallel plate model used in the experiments had a uniform gap throughout [see Fig. 1(*b*)]. This simulated an isotropic and homogeneous soil mass. It is possible, however, to simulate nonhomogeneous and anisotropic mass by appropriately varying the gap between the plates. The permeability of the model for use in Eqs. 1 and 2 was obtained from the following relation (11,16,31):

$$k_m = k_x = k_y = \frac{b^2 \rho g}{3\mu} \quad \text{(2c)}$$

in which: b = half width of gap; ρ = density of fluid; g = gravitational constant; and μ = viscosity of the fluid.

FINITE DIFFERENCE REPRESENTATION

The finite difference ADEP form corresponding to Eq. 1*a* is

$$h_{i,j,t+1} = h_{i,j,t} + \beta_x \left(\frac{h^2_{i-1,j,t+1} - h^2_{i,j,t+1}}{\Delta x1} - \frac{h^2_{i,j,t} - h^2_{i+1,j,t}}{\Delta x2} \right)$$

$$+ \beta_y \left(\frac{h^2_{i,j+1,t+1} - h^2_{i,j,t+1}}{\Delta y1} - \frac{h^2_{i,j,t} - h^2_{i,j-1,t}}{\Delta y2} \right) \quad \text{(3a)}$$

in which $\beta_x = k_x \Delta t/[n(\Delta x1 + \Delta x2)]$ and $\beta_y = k_y \Delta t/[n(\Delta y1 + \Delta y2)]$. Eq. 3*a*, at time level $t + 1$, reduces to a quadratic form:

$$A h^2_{i,j,t+1} + B h_{i,j,t+1} + C = 0 \quad \text{(3b)}$$

in which Δx and Δy = spatial intervals; Δt = time interval; and A, B, C = known constants at $(t + 1)$, and are functions of β_x, β_y, and known heads at t

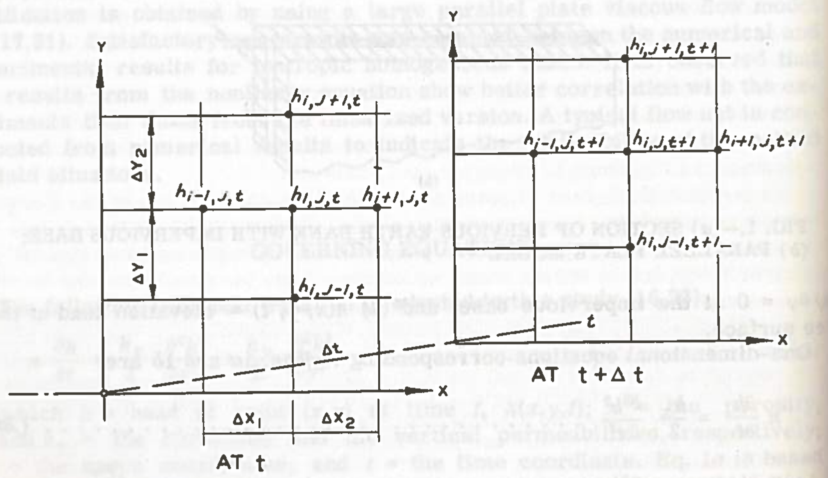


FIG. 2.—FINITE DIFFERENCE APPROXIMATION FOR ADEP

and $t + 1$. The index symbols are shown in Fig. 2.

The finite difference form corresponding to the linearized Eq. 1*b* is

$$h_{i,j,t+1} = h_{i,j,t} + \overline{\beta}_x \left(\frac{h_{i-1,j,t+1} - h_{i,j,t+1}}{\Delta x1} - \frac{h_{i,j,t} - h_{i+1,j,t}}{\Delta x2} \right)$$

$$+ \overline{\beta}_y \left(\frac{h_{i,j+1,t+1} - h_{i,j,t+1}}{\Delta y1} - \frac{h_{i,j,t} - h_{i,j-1,t}}{\Delta y2} \right) \quad \text{(4)}$$

in which $\overline{\beta}_x = k_x \overline{h}/[0.5n(\Delta x1 + \Delta x2)]$, and $\overline{\beta}_y = k_y \overline{h}/[0.5n(\Delta y1 + \Delta y2)]$. Similar forms were obtained for the one-dimensional Eqs. 2*a* and 2*b*.

In Eqs. 3 and 4, two time levels are used. At each of the two time levels, only one head from each direction is included. If a proper choice of an initial starting point is made, e.g., at the upstream face where $f(t)$ is prescribed for all times, then $h_{i,j,t+1}$ is the only unknown and can be computed explicitly. Eqs. 3 and 4 are sequentially applied point by point, either in the x and y directions. The accuracy of this procedure is improved by adopting the point-by-point sequence in alternate directions. It has been found that the ADEP is

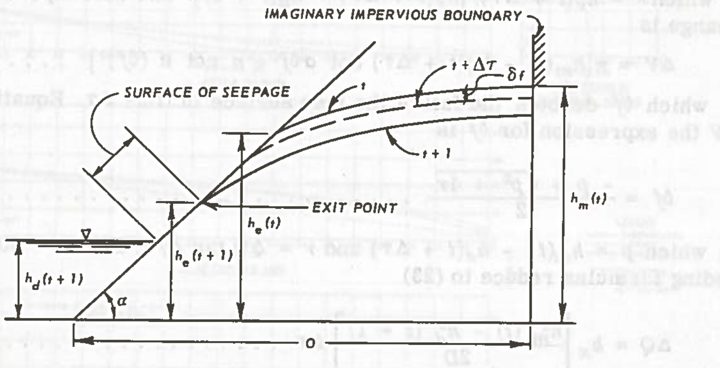


FIG. 3.—LOCATION OF SURFACE OF SEEPAGE

more suitable and computationally stable compared to some other finite difference schemes and can also be extended for three-dimensional flow (2).

The solution usually starts at time $t = 0$ when the head distribution in the flow domain is specified, as in the boundary condition 1. The solutions for increasing times are then propagated by using the recurrence Eq. 3 or Eq. 4. These equations are easy to program for the computer.

Rise of External Water Level.—The solution for this condition proceeds as described previously with the applied heads at the entrance known at all time levels according to the boundary condition 2. The approximate location of the free surface at a given time is obtained on the basis of the known heads at that time. It is achieved by locating in the flow domain, those points at which the computed head is equal to the elevation head. A linear interpolation scheme is used to satisfy this condition stated in the boundary condition 4. Thus, the free surface is located at any selected time level.

Fall, or Drawdown, of External Water Level.—This case involves additional complexities due to the occurrence of the surface of seepage (see Fig. 3). The surface of seepage arises because the fall of the free surface lags behind the fall of the external water level. Consequently, the surface exits the entrance face at a point higher than the external water level. As the seepage forces near the entrance are influenced by the location of the exit point, it is necessary to compute it for all time levels. Furthermore, determination of the exit point is necessary for imposing appropriate modified upstream heads.

Pavlovsky's method of fragments (16,18) or the method of permeable membrane (24) may be used to locate the point of exit. The method of fragments, which was also recently used by Dvinoff (13), is employed herein and is explained briefly. Fig. 3 shows a section of the bank at a typical instant of time, $t+1$, during drawdown. To apply the method, the time interval Δt from t to $t+1$, is divided into a number of small time intervals $\Delta\tau$. For instance, Δt = 100 sec used in this study was divided into 5,000 small $\Delta\tau$. The quantity of fluid out of the upstream face is equal to the amount of fluid contained between the free surfaces at two time levels $\Delta\tau$ apart. Therefore, the flow per unit length out, assuming it to be essentially horizontal, is

$$\Delta Q = -k_x\,[h_e(t+\Delta\tau) - h_d(t+1)]\tan\alpha\,(1+\log\lambda)\,\Delta\tau \quad \ldots\ldots \quad (5a)$$

in which $\lambda = h_e(t+\Delta\tau)/[h_e(t+\Delta\tau) - h_d(t+1)]$. The corresponding volume change is

$$\Delta V = n[h_m(t) - h_e(t+\Delta\tau)\cot\alpha\,\delta f + n\cot\alpha\,(\delta f)^2] \quad \ldots\ldots \quad (5b)$$

in which δf denotes the fall of the free surface in time $\Delta\tau$. Equating ΔQ and ΔV the expression for δf is

$$\delta f = \frac{-p + \sqrt{p^2 + 4r}}{2} \quad \ldots\ldots \quad (5c)$$

in which $p = h_m(t) - h_e(t+\Delta\tau)$ and $r = \Delta Q\tan\alpha/n$. For $\alpha = 90°$, the preceding formulas reduce to (23)

$$\Delta Q = k_x\left[\frac{h_m^2(t) - h_d^2(t+1)}{2D}\right]\Delta\tau \quad \ldots\ldots \quad (6a)$$

$$\Delta V = n\,D\,\delta f \quad \ldots\ldots \quad (6b)$$

$$\text{and } \delta f = \frac{\Delta Q}{nD} \quad \ldots\ldots \quad (6c)$$

in which D denotes the distance between the entrance toe and the location of the maximum head (see Fig. 3). In using the foregoing procedure, an impervious boundary is needed. For the model which represents a long riverbank, no such physical boundary is generally available. Therefore, an imaginary impervious boundary is established at each time level. This is done by locating the point of maximum head $h_m(t)$ at the previous time, and using the vertical through that point as the required impervious boundary.

Once the fall of the surface δf corresponding to $\Delta\tau$ is computed, the location of the exit point is obtained from the recurrence relation:

$$h_e(t_i + \Delta\tau) = h_e(t_i) - \delta f_i \quad \ldots\ldots \quad (7)$$

in which t_i assumes values from t to $t+1$, which includes, e.g., 5,000 iterations for Δt = 100 sec. The last value, $h_e(t+1)$, gives the exit head for the current time, $t+1$.

The entrance boundary heads are now imposed as

$$\left.\begin{aligned} h(x, x\tan\alpha, t+1) &= h_d(t+1) \text{ for points below } h_d(t+1)\\ h(x, x\tan\alpha, t+1) &= \text{elevation head along the surface of seepage}\\ h(x, x\tan\alpha, t+1) &= \frac{[h_e(t+1) + h_m(t)]}{2} \text{ for points above } h_e(t+1)\end{aligned}\right\} \quad (8)$$

The last condition is arbitrarily chosen.

Eqs. 3 or 4 is now applied and the procedure repeated until desired drawdown level is attained.

For the constant head time zone, between the time of maximum rise and the time when the fall starts, (see Fig. 4) Eqs. 3 or 4 are used in the same manner as for the rise condition.

Boundary Conditions at Downstream Face.—The level of fluid at the tail or exit end was always maintained zero. The foregoing procedure for locating

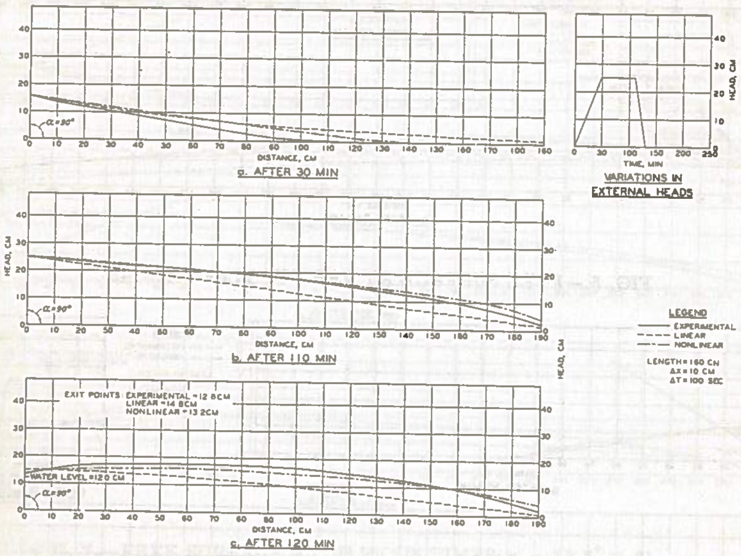


FIG. 4.—FREE SURFACE AT VARIOUS TIMES: $\alpha = 90°$

the surface of seepage was not employed for the exit face because the variations in the exit point for the model of large length was very slow. Instead, an approximation was made, according to which, zero head was assumed at a distance of one Δx outside the exit face, and a linear head variation from this point to the point one Δx inside from the exit face.

EXPERIMENTAL RESULTS

A number of experiments were conducted by the U.S. Army Engineer Waterways Experiment Station with a viscous flow model, about 350 cm long and 50 cm high (11). Only results from four tests are reported herein. Silicone fluid, which is stable under the influence of temperature was used, and the level in

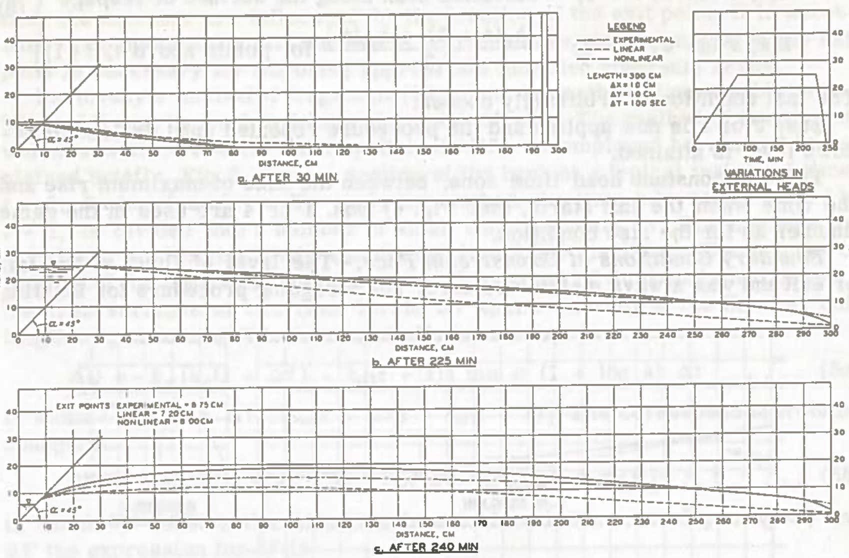


FIG. 5.—FREE SURFACE AT VARIOUS TIMES: α = 45°

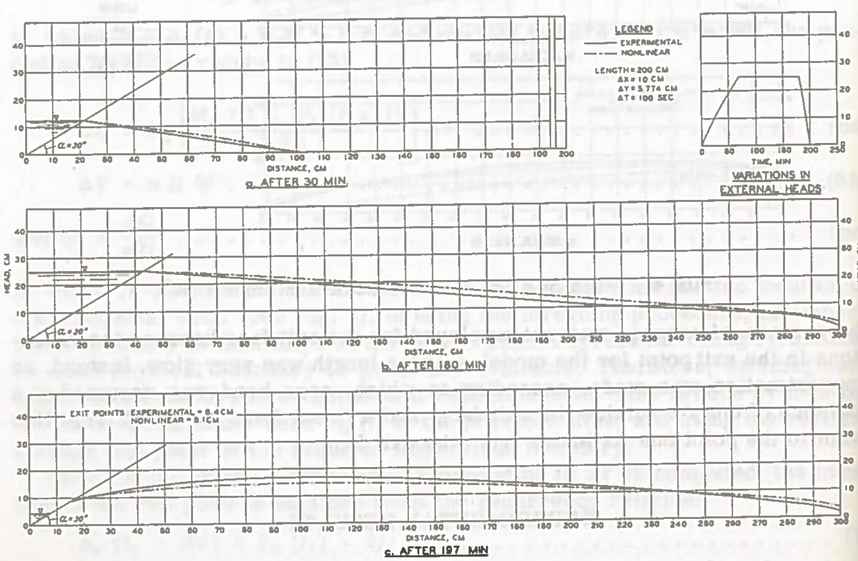


FIG. 6.—FREE SURFACE AT VARIOUS TIMES: α = 30°

the reservoir of the model was varied by pumping the fluid at desired rates. A special mechanical device was installed to allow the fall or drawdown of the fluid level. After a desired height was reached, the free surface was observed under a constant head till the free surface was relatively stable. Then the fluid level was allowed to fall. Typical variations in the fluid level are shown in Figs. 4, 5, 6, and 7.

In the first test, the length of the rectangular plates, α = 90°, was about 190 cm with an average width of gap of 0.2 cm (see Fig. 4). One-dimensional equations were used for this case. Three different plates with upstream slopes of 45°, 30°, and 18.5° (1:3), and the same length of 300 cm, with an average gap width of 0.17 cm were used for the other three tests (see Figs. 5, 6, and 7). The permeability of the model was computed from Eq. 2c in which the values

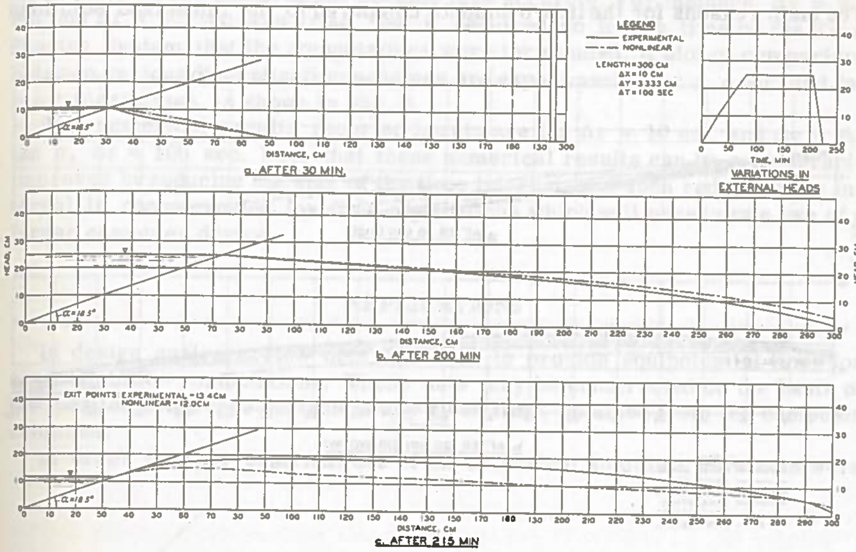


FIG. 7.—FREE SURFACE AT VARIOUS TIMES: α = 18.5° (1:3)

of various quantities were: ρ = 0.97 gm per cu cm; g = 980 cm per sec squared; μ = 9.8 stokes. The porosity of the model, n, was assumed as unity.

COMPARISON AND ASSESSMENT OF RESULTS

Experimental results for the four tests are plotted in Figs. 4, 5, 6, and 7, in comparison with the numerical solutions. Each of these figures contain three typical plots: The first during the rising water level; the second at the steady state condition; and the third during the drawdown. The results from the linearized equations give satisfactory comparison at earlier times, but generally, only in the regions in the vicinity of the entrance face. These results showed poor correlation with the experiments in the majority of the flow domain, at higher time levels, and for the fall (drawdown) conditions. Dvinoff (13) used the linearized equation and compared the numerical results with experiments with two viscous flow models with upstream slopes of 45° and 1:2, and lengths equal to 24 in. and 30 in., respectively. The linearized solutions gave satisfactory comparison in the flow zones in the vicinity of the entrance face. In a previous study (4), closed form analytical solutions for the linearized equation were compared with experiments from models with upstream slopes of 90°, 45° and 22.5°, and similar lengths. Satisfactory comparisons were obtained in the vicinity of the entrance face, but the two results showed significant differences in other regions of the flow domain.

In the case of long riverbanks and embankments, it is important that the solutions are satisfactory for the entire flow domain. Particularly, the designer is interested in seepage forces near both the entrance and the exit faces. The nonlinear form of the equation presented herein showed improved agreement with the experiments for the entire flow domain and for all times. The main reasons for the improvements compared to the linearized solutions

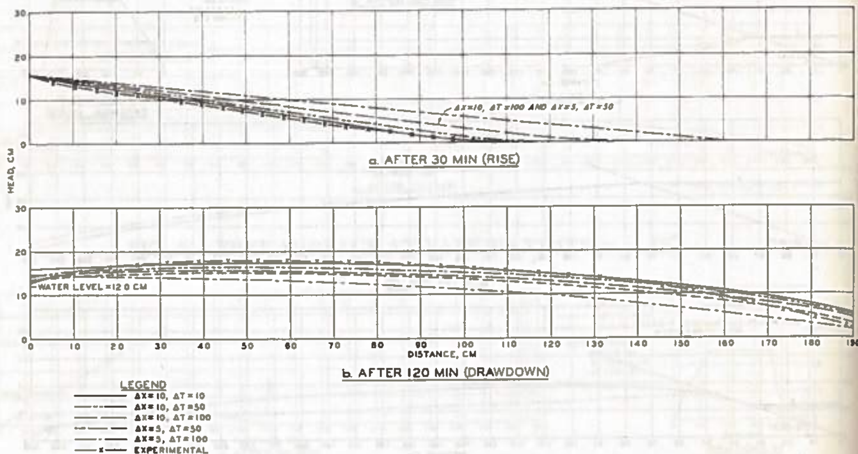


FIG. 8.—EFFECT OF DISCRETIZATION SCHEMES ON NUMERICAL SOLUTIONS

may be due to the higher order nature of the head distributions inherent in the nonlinear solution (23) and the influence of the impervious boundary (11).

Figs. 4 and 5 for α = 90° and α = 45° are plotted to show comparison between nonlinear and linearized solutions, and the experimental results. As the nonlinear solutions are found better and preferable, Figs. 6 and 7 for α = 30° and 18.5° slopes are plotted to show only the nonlinear and the experimental results.

Numerical Discretization and its Effects on Solutions.—The finite difference mesh size for the results reported in Figs. 4, 5, 6, and 7 was Δx = 10 cm, $\Delta y = \Delta x \tan \alpha$ and Δt = 100 sec. The extent of the vertical domain included as flow region was determined on the basis of the magnitudes of the head h developed in the flow domain. When computed, h at a node point was very small, the cycle of analysis was terminated and the next cycle was started. It was observed that the heads became negligible at a distance from the base, of approximately four to six times the maximum head in the reservoir.

A Time Sharing system connected with a GE 400 computer system was used for all computations reported herein. It was found that the numerical solutions gained accuracy as the time interval was reduced. For example, for the same Δx and Δy, the solution gained about 5 %-10 % accuracy with the reduction in Δt from 100 sec to 50 sec. For Δt = 10 sec, the numerical solution gained further accuracy and showed very close agreement with the experiments. The computation time, however, increased with decreasing Δt. For instance, the computation times for Δt = 100 sec, 50 sec, and 10 sec for the same Δx = 10 cm, were in the approximate ratios of 1:2:10. The approximate time for computing free surface for six time levels, for Δx = 10 cm and Δt = 100 sec, was 30 sec. No significant accuracy was observed for decrease in the spatial subdivision and it required much higher computer times. Often, the results from finer space subdivisions became less accurate. For instance, Δx = 2.5 cm and Δt = 50 sec was tried, but it required so long a time on the Time Sharing System that the computations were terminated. A plot of comparison between various discretization schemes and experiments, for α = 90° and for two typical times, is shown in Fig. 8.

The numerical results reported herein are for Δx = 10 cm, and $\Delta y = \Delta x \tan \alpha$, Δt = 100 sec. Note that these numerical results can be considerably improved by reducing the size of the time interval, and such reduced time interval is recommended for field applications which will necessitate use of a larger computer device.

APPLICATIONS

In design analysis, flow nets are used to provide equipotential lines for slope stability computations. These nets may be constructed on the basis of the location of the free surface at any given time, obtained from the computer solutions.

In order to show practical use of the numerical solutions, flow nets were

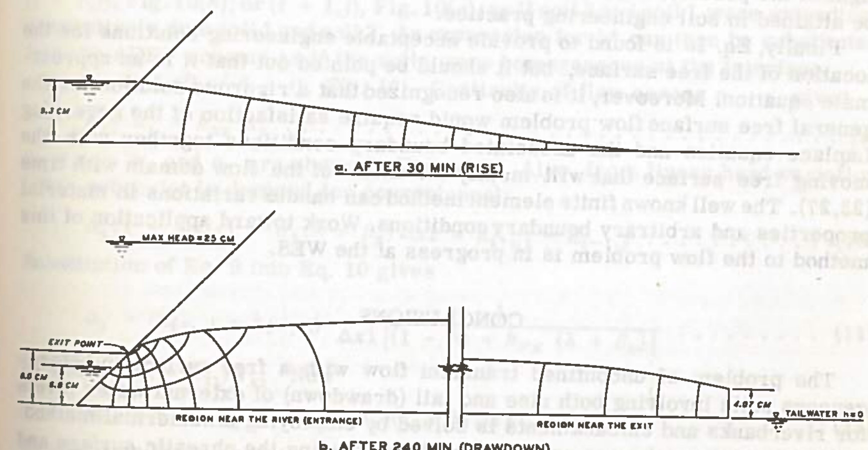


FIG. 9.—TYPICAL FLOW NETS: α = 45°

constructed (see Fig. 9) for two typical time levels for $\alpha = 45°$ on the basis of the nonlinear solutions. Similar flow nets can be constructed for any other time levels. Extension of the computer program for field conditions is straightforward, and such programs independent of units and in nondimensional form are prepared.

Anisotropic soil conditions can be incorporated in the formulation by specifying different permeabilities k_x and k_y in the x and y directions. Nonhomogeneous continua require fulfillment of additional continuity of flow conditions at the interfaces of different materials (1). Such a formulation is presented in Appendix I. The preceding formulations are also applicable to the case of sudden drawdown.

COMMENTS

A number of error sources are possible in the experimental results. Often, a particular model did not closely fit, and some leakage of fluid was observed. Uniform gap between the plates was expected and used in the numerical analysis. However, due to many reasons such as nonuniform thickness of plates themselves, their large length and manufacturing errors, the gap was not uniform. The friction at the base was neglected as well as the capillary effects. The silicone fluid was found relatively stable, still some variations in its viscosity and density were inevitable with different room temperatures. All these, and other reasons might have influenced to an extent the experimental results. For instance, in the case of $\alpha = 45°$, the nonlinear numerical curve at steady state condition fell below the experimental curve, whereas in all other tests it generally fell above. It was further observed that the numerical solutions do not provide as good a correlation for slope of 1:3, as they do for higher slope angles (see Fig. 7). Due to the assumption of linear head variation near the exit face, the two results showed comparatively larger differences near that face. Overall, the numerical solutions are considered satisfactory in the light of the possible experimental errors and the degree of precision that can be attained in soil engineering practice.

Finally, Eq. 1*a* is found to provide acceptable engineering solutions for the location of the free surface, but it should be pointed out that it is an approximate equation. Moreover, it is also recognized that a rigorous solution for the general free surface flow problem would require satisfaction of the governing Laplace equation and the associated boundary conditions together with the moving free surface that will modify the extent of the flow domain with time (23,27). The well known finite element method can handle variations in material properties and arbitrary boundary conditions. Work toward application of this method to the flow problem is in progress at the WES.

CONCLUSIONS

The problem of unconfined transient flow with a free surface in heterogeneous soils involving both rise and fall (drawdown) of external water levels for riverbanks and embankments is solved by employing a numerical method. Special iterative schemes are employed for locating the phreatic surface and the surface of seepage. Satisfactory correlation is obtained between the numerical and experimental results. A nonlinear form of the flow equation showed

promise of excellent agreement with refinement of the time-wise subdivision. It is further showed that the nonlinear equation proposed in this study gives better results than the linearized equation used previously. A flow net is constructed from typical numerical results to indicate the practical utility of the method. It is believed that the numerical method proposed herein can be conveniently adopted for field-design applications.

ACKNOWLEDGMENT

The tests described and the resulting data presented herein are obtained from research conducted under the project on seepage studies in the Mississippi Riverbanks supported by the U.S. Army Engineer Division, Lower Mississippi Valley of the U.S. Army Corps of Engineers by the Waterways Experiment Station, Vicksburg, Mississippi. Useful comments and suggestions by R. I. Kaufman and A. Dvinoff, LMVD, and R.S. Sandhu, Ohio State University, are gratefully acknowledged. The permission granted by the Chief of Engineers to publish this information is appreciated.

APPENDIX I.—NONHOMOGENEOUS SOILS

Fig. 10 shows an inclined interface between two soils with different permeabilities. For convenience, an inclined interface is treated as a combination of horizontal and vertical interfaces (1,11,13).

Vertical Interface v - v.—As shown in Fig. 10, v - v can occur either to the left or to the right of the node point (i,j). Linear head variation is assumed between nodes $(i - 1,j)$, (i,j), and $(i + 1,j)$. It is found necessary as a numerical expedient, to establish a fictitious head h^* that represents head at node points $(i - 1,j)$, Fig. 10(b); or $(i + 1,j)$, Fig. 10(c), as if soil 2 and soil 1 were extending, respectively, into soil 1 and soil 2. An expression for h^* can then be substituted into the ADEP scheme as if the soils were homogeneous at the interface.

v - v to Left of $(i - j)$, Fig. 10.—Continuity of flow across v - v, gives

$$k_{x1}\,\alpha_1 = k_{x2}\,\alpha_2 \qquad (9)$$

in which α_1 and α_2 are shown in Fig. 10(b). Also, from linear head variation (time subscript is dropped for convenience):

$$\alpha_1(1 - \lambda)\,\Delta x1 + \alpha_2(\lambda + \beta_x)\,\Delta x1 = h_{i+1,j} - h_{i-1,j} \qquad (10)$$

Substitution of Eq. 9 into Eq. 10 gives

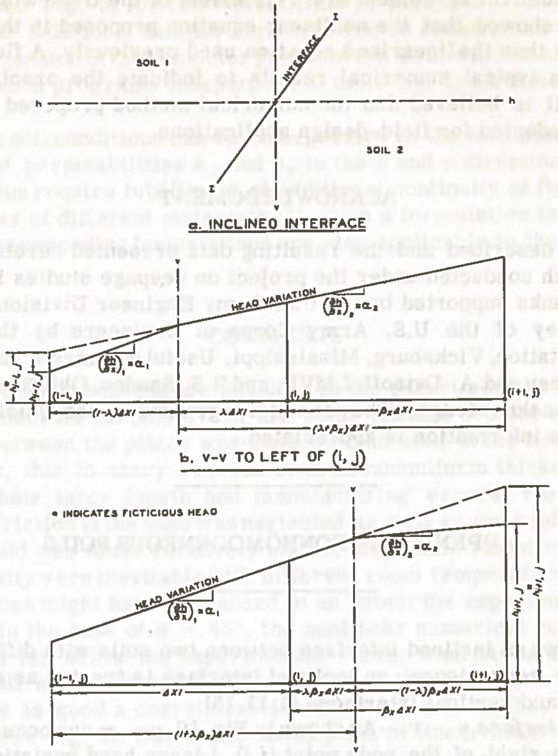
$$\alpha_1 = (h_{i+1,j} - h_{i-1,j})\,\frac{1}{\Delta x1\left[(1 - \lambda) + k_{rx}\,(\lambda + \beta_x)\right]} \qquad (11)$$

in which $k_{rx} = k_{x1}/k_{x2}$. Now

$$h^*_{i-1,j} = h_{i-1,j} + (\alpha_1 - \alpha_2)(1 - \lambda)\,\Delta x1 \qquad (12)$$

Substituting for α_1 and α_2, Eq. 12 becomes

$$h^*_{i-1,j} = h_{i-1,j} + (h_{i+1,j} - h_{i-1,j})\,\gamma_{x1} \qquad (13a)$$



a. INCLINED INTERFACE

b. V-V TO LEFT OF (I, J)

c. V-V TO RIGHT OF (I, J)

FIG. 10.—NONHOMOGENEOUS MATERIAL

in which $$\gamma_{x1} = \frac{(1 - k_{\gamma x})(1 - \lambda)}{(1 - \lambda) + k_{\gamma x}(\lambda + \beta_x)} \quad (13b)$$

In a similar manner, for $v - v$ to the right of (i,j), (see Fig. 10), the fictitious head is

$$h^*_{i+1,j} = h_{i+1,j} + (h_{i+1,j} - h_{i-1,j})\,\gamma_{x2} \quad (14a)$$

in which $$\gamma_{x2} = \frac{(1 - k_{\gamma x})(1 - \lambda)}{(1 + \lambda\beta_x) + k_{\gamma x}(1 - \lambda)\beta_x} \quad (14b)$$

Horizontal Interface h - h.—Following the similar procedure for $h - h$ above and below the node point (i,j), the fictitious heads are, respectively

$$h^*_{i,j-1} = h_{i,j-1} + (h_{i,j+1} - h_{i,j-1})\,\gamma_{y1} \quad (15)$$

and $$h^*_{i,j+1} = h_{i,j+1} + (h_{i,j+1} - h_{i,j-1})\,\gamma_{y2} \quad (16a)$$

in which $$\gamma_{y1} = \frac{(1 - k_{\gamma y})(1 - \lambda)}{(1 - \lambda) + k_{\gamma y}(\lambda + \beta_y)} \quad (16b)$$

$$\gamma_{y2} = \frac{(1 - k_{\gamma y})(1 - \lambda)}{(1 + \lambda\beta_y) + k_{\gamma y}(1 - \lambda)\beta_y} \quad (16c)$$

$$k_{\gamma y} = \frac{k_{y1}}{k_{y2}} \quad (16d)$$

The expressions for h^* can now be substituted in the ADEP scheme to account for an interface between two materials.

APPENDIX II.—REFERENCES

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APPENDIX III.—NOTATION

The following symbols are used in this paper:

A = a constant;
B = a constant;
b = half width of gap in model;
C = a constant;
D = distance between entrance toe and point of maximum head;
$f(t)$ = function specifying variation of water level;
g = gravitational constant;
h = head;
h_d = reservoir head;
h_e = exit head;
h_m = maximum head;
$\bar{h}$ = mean head in reservoir;
h^* = fictitious head;
k = permeability;
k_m = permeability of model;

k_r = permeability ratio;
k_x = permeability in x-direction;
k_y = permeability in y-direction;
n = porosity;
p = a factor;
Q = quantity of flow;
r = a factor;
t = time;
V = fluid volume;
x = coordinate;
y = coordinate;
α = angle, gradient;
β = factor;
γ = factor, density;
Δt = time interval;
Δx = interval in x-direction;
Δy = interval in y-direction;
$\Delta\tau$ = incremental fractions of Δt;
δf = incremental change in free surface;
λ = factor;
μ = viscosity; and
ρ = mass density.