VOL.97 NO.SM11. NOV. 1971

JOURNAL OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION

PROCEEDINGS OF
THE AMERICAN SOCIETY
OF CIVIL ENGINEERS

AMERICAN SOCIETY OF CIVIL ENGINEERS FOUNDED 1852

VOL.97 NO.SM11. NOV. 1971

JOURNAL OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION

PROCEEDINGS OF THE AMERICAN SOCIETY OF CIVIL ENGINEERS

CONTENTS

Papers — Page

DISCUSSION

Proc. Paper 8484

(over)

This Journal is published monthly by the American Society of Civil Engineers. Publications office is at 345 East 47th Street, New York, N.Y. 10017. Address all ASCE correspondence to the Editorial and General Offices at 345 East 47th Street, New York, N.Y. 10017. Allow six weeks for change of address to become effective. Subscription price is $16.00 per year with discounts to members and libraries. Second-class postage paid at New York, N.Y. and at additional mailing offices. SM.

TECHNICAL NOTE

Proc. Paper 8511

INFORMATION RETRIEVAL

The key words, abstract, and reference "cards" for each article in this Journal represent part of the ASCE participation in the EJC information retrieval plan. The retrieval data are placed herein so that each can be cut out, placed on a 3 x 5 card and given an accession number for the user's file. The accession number is then entered on key word cards so that the user can subsequently match key words to choose the articles he wishes. Details of this program were given in an August, 1962 article in CIVIL ENGINEERING, reprints of which are available on request to ASCE headquarters.

[a] Discussion period closed for this paper. Any other discussion received during this discussion period will be published in subsequent Journals.

8497 HEAVE AND LATERAL MOVEMENTS DUE TO PILE DRIVING

KEY WORDS: Clays; **Construction**; **Foundations**; Heaving; Pile foundations; **Settlement (structural)**; **Soil mechanics**

ABSTRACT: Vertical and lateral displacements of soil and driven piles occur during pile driving in certain types of soils. The occurrence of vertical heave often significantly adds to the cost of a pile foundation. The undetected occurrence of heave of piles and foundation soils can lead to ultimate failure of a pile foundation. A case history study of 13 pile foundations was undertaken. Mechanisms of soil and pile displacements are proposed and the effects of various factors on the displacement phenomena are analyzed. The principal factors which affect the mode and magnitude of soil and pile displacement are: (1) The characteristics of the soil into which the piles are driven; (2) the characteristics of the piles themselves; (3) the sequence of pile driving; and (4) the overall geometry of the pile foundation. An approximate method of prediction of the magnitude of soil and pile heave is presented.

REFERENCE: Hagerty, Joseph D., and Peck, Ralph B., "Heave and Lateral Movements Due to Pile Driving," **Journal of the Soil Mechanics and Foundations Division**, ASCE, Vol. 97, No. SM11, Proc. Paper 8497, November, 1971, pp. 1513-1532

8510 SCALE AND BOUNDARY EFFECTS IN FOUNDATION ANALYSIS

KEY WORDS: **Bearing capacity**; Boundary conditions; **Coefficients**; **Computation**; **Failure**; **Foundations**; **Models**; **Plasticity**; **Sands**; Scale effect; **Soil mechanics**; **Stability**

ABSTRACT: The bearing capacity of rough strip footings on sand is analyzed using numerical integration of the plasticity equations. The influence of different boundary conditions below the footing is studied for the zero surcharge case, and bearing capacity coefficients are calculated for various assumed settlements at failure. A pressure dependent solution is described which permits calculation of the variation of ϕ with stress level in the failure zone. The results are compared with existing trial failure surface solutions and with experimental values. It is concluded that good agreement can be obtained between theoretical results which assume a trapped elastic wedge beneath the footing, and model results related to average ϕ-values from triaxial tests in the normal range of cell pressures. The lack of agreement with field results and the computed variation of ϕ in the failure zone imply that this relationship is empirical. A procedure is outlined for relating bearing capacity of a given-sized footing to the initial density of a sand using ϕ verses pressure results from plane strain shear tests on sand at the same density.

REFERENCE: Graham, James, and Stuart, John Gordon, "Scale and Boundary Effects in Foundation Analysis," **Journal of the Soil Mechanics and Foundations Division**, ASCE, Vol. 97, No. SM11, Proc. Paper 8510, November, 1971, pp. 1533-1548

8522 ANALYSIS OF LOAD-BEARING FILLS OVER SOFT SUBSOILS

KEY WORDS: **Bearing capacity**; Bearing values; Cohesionless soils; Cohesive soils; **Deformation modulus**; Design criteria; Elastic theory; **Fills**; **Footings**; Foundation bearing tests; Poisson ratio; **Settlement (structural)**; Shear failure; **Soil mechanics**; **Stress distribution**; Stress-strain curves; Subsoil

ABSTRACT: The protection afforded soft subsoils by the stress-distributing characteristics of load-bearing fills is examined. Using recently developed finite element techniques of analysis, including nonlinear soil properties, a complete stress-deformation solution of a fill-subsoil system is obtained. The proposed method is first examined by applying it to three problems, one hypothetical and two actual load test case histories. The method is then used to study the stress distributing characteristics of granular fills overlying clay subsoils. As would be anticipated, the stiffer the fill relative to the foundation soil, the greater the reduction in stress transmitted to the underlying soil. Although the complexity of the load-bearing fill problem is such that simple charts cannot be prepared to indicate fill-subsoil interface stresses for different soil conditions and geometries, it is shown that approximate values of maximum vertical and shear stress can be obtained by use of an easily applied formula for modulus characterization and a simple geometrical function.

REFERENCE: Mitchell, James K., and Gardner, William S., "Analysis of Load-Bearing Fills over Soft Subsoils," **Journal of the Soil Mechanics and Foundations Division**, ASCE, Vol. 97, No. SM11, Proc. Paper 8522, November, 1971, pp. 1549-1571

Journal of the

SOIL MECHANICS AND FOUNDATIONS DIVISION

Proceedings of the American Society of Civil Engineers

APPROXIMATE SOLUTION TO FLOW PROBLEM UNDER DAMS

By Demetrius G. Christoulas,[1] M. ASCE

INTRODUCTION

For a hydraulic structure resting on a homogeneous and isotropic pervious stratum, the two-dimensional flow problem can be considered as one of conformal representation. In 1922, Pavlovsky proposed the use of the Christoffel-Schwarz transformation for the solution of this problem in the following way (4,8). Flow region $z = x + iy$ is represented on the plane of complex potential $\omega = \phi + i\Psi$ by rectangle ABDC. Assuming that the contour of the flow region is composed of horizontal and vertical straight-line segments, the conformal mapping of polygons ω and z is achieved on half plane ζ through the following Christoffel-Schwarz transformations:

$$\omega = M \int \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - m^2 \zeta^2)}} + N = M\, F(\zeta, m) + N \quad \ldots\ldots\ldots (1)$$

$$z = M' \int \frac{\Pi\,(\zeta - a_i)}{\zeta^2 - \dfrac{1}{m^2}\ \sqrt{\zeta^2 - 1}\ \Pi\,(\sqrt{\zeta - c_i})}\, d\zeta + N' \quad \ldots\ldots\ldots (2)$$

in which Π denotes the product. Function $F(\zeta, m)$ is an elliptic integral of the first kind and it can be computed easily with the use of tables. The second function is generally a hyperelliptic integral and parameters m, a_i, and c_i are not known. Thus the solution can be achieved only for simple flow regions.

Due to the difficulty of an exact solution of the problem, Pavlovsky proposed the well-known method of fragments. The fundamental assumption of this method is that the vertical lines through the ends of the cut-offs are equipotential lines. Thus flow region z and its image on plane ω are divided into simple orthogonal fragments. The Christoffel-Schwarz transformation can, therefore, be applied, and it easily produces the solution to the problem

Note.—Discussion open until April 1, 1972. To extend the closing date one month, a written request must be filed with the Executive Director, ASCE. This paper is part of the copyrighted Journal of the Soil Mechanics and Foundations Division, Proceedings of the American Society of Civil Engineers, Vol. 94, No. SM11, November, 1971. Manuscript was submitted for review for possible publication on March 22, 1971.

[1] Hydraulic Consulting Engr., Athens, Greece.

providing that the most complex functions resulting are elliptic integrals of the first kind. Pavlovsky's assumption is more exact as the ratios, S/T, increase, in which S represents the depth of the cut-offs and T, the depth of the pervious stratum.

Besides Pavlovsky's method, the approximate methods proposed by Khosla (6), Melechenko-Filchakov (1,8), and Chugaev (9) are applied. These methods do not aim at a complete solution of the flow problem, but at the computation of the proper quantities necessary in the design of hydraulic structures. These methods, and especially Chugaev's, present a simple and rapid application.

The present work introduces a new approximate analytical method which is essentially an improvement of Pavlovsky's method.

FUNDAMENTAL ASSUMPTION OF PROPOSED METHOD

It is assumed that the contour of the flow region is composed of horizontal and vertical straight-line segments [Fig. 1(a)]. The number of cut-offs is not

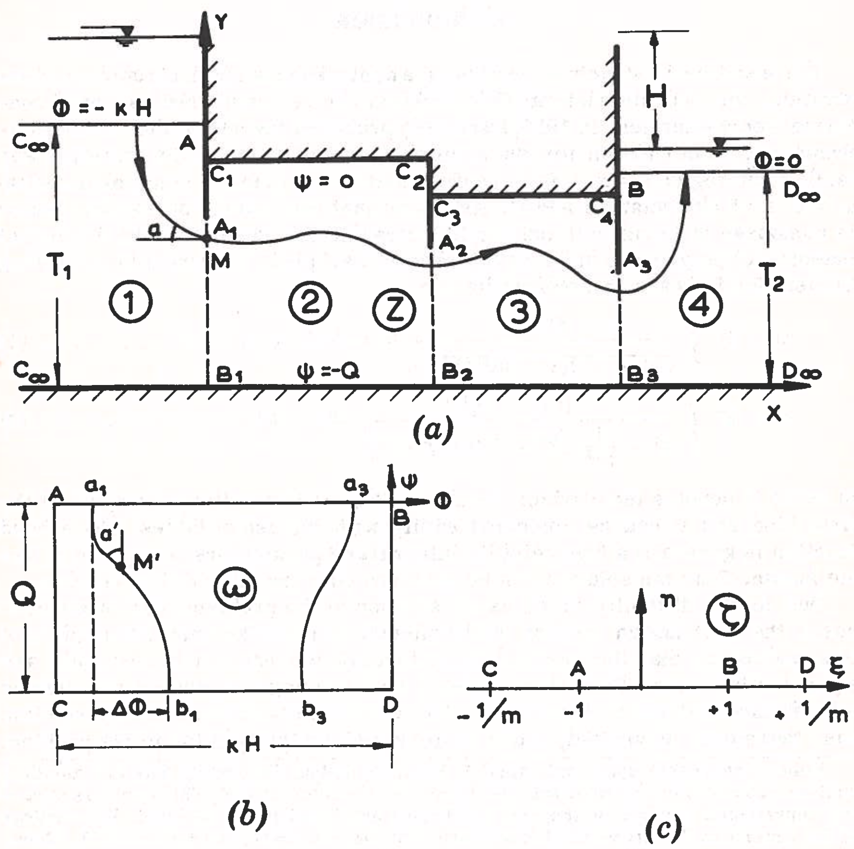

FIG. 1.—FLOW REGION AND ITS IMAGES ONTO PLANES W AND ζ

restricted. Herein, the problem is handled in the same way as did Pavlovsky, i.e., the flow region is divided into simple orthogonal fragments by the vertical lines through the ends of the cut-offs. However, these lines, AB(A_1 B_1, A_2 B_2, etc), are not equipotential, as Pavlovsky had assumed them to be, and, therefore, their images on plane ω are not straight lines but curvilinear segments ab [Fig. 1(b)]. An effort has been made to find a family of curves approaching, as much as possible, curves ab. As representation ω - z is conformal, slope angle a of the streamline passing from point M is equal to angle a' between image curve ab and the Ψ axis, at corresponding point M'. If point M moves from A to B, slope angle a varies in a typical way. Indeed, this angle vanishes at the vicinity of point A, increases quickly to a maximum value, and vanishes at point B. Also, angle a assumes limited values. Consequently, the same holds true for angle a'. This angle varies in the described typical way and assumes limited values. It is further observed that the ratio $|\Delta\phi|/Q$ assumes limited values too. On the basis of these observations it can be assumed that the shape of curves ab is determined only by parameters $\Delta\phi$ and Q, and that all the special characteristics of the flow region have no other ef-

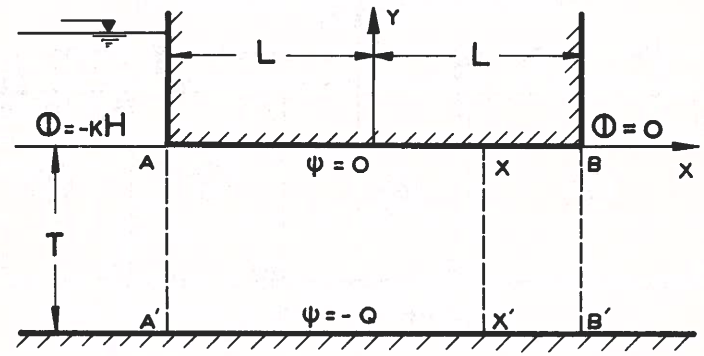

FIG. 2.—BASIC FLOW REGION

fect on the shape of these curves. Thus a special simple flow region can be used and the corresponding family of curves ab can be generally applied. Such a suitable simple flow region is shown in Fig. 2. The simplicity of this region facilitates computation. Moreover, this region is the limit case of a dam with small depth cut-offs, i.e., the case in which a better approximation to the real shape of the curves ab is needed, because of the great $\Delta\phi$. In the case of a dam with great depth cut-offs any assumption about the shape of curves ab will always be satisfactory, because of the small $\Delta\phi$. The exact solution of the problem in Fig. 2 is given (8) by

$$\left.\begin{aligned} z = x + iy &= \frac{T}{\pi} \ln \frac{1 + m \operatorname{sn}\left(\dfrac{2K\omega}{\kappa H} + K\right)}{1 - m \operatorname{sn}\left(\dfrac{2K\omega}{\kappa H} + K\right)} \\ Q &= \frac{K'(m)}{2K(m)} \kappa H \qquad m = \tanh \frac{\pi L}{2T} \end{aligned}\right\} \qquad (3)$$

For a given flow region with cut-offs [Fig. 1(a)] characterized by total head H and seepage discharge Q, there is always a suitable flow region in Fig. 2 characterized by the same quantities H and Q. This flow region will hereafter be called the basic flow region. For a vertical segment XX' moving from AA' to BB' (Fig. 2), the quotient $\Delta\phi/Q = \left(\phi_x - \phi_{x'}\right)/Q$ varies continuously between minimum value $\left(\phi_A - \phi_{A'}\right)/Q$ and maximum $\left(\phi_B - \phi_{B'}\right)/Q$. It is evident that $-\left(\phi_A - \phi_{A'}\right) = \left(\phi_B - \phi_{B'}\right) > 0$. The maximum potential difference $|\phi_A - \phi_B|$ in Fig. 1(a) is observed at the upstream or the downstream cut-off. As a rule, this maximum difference will be smaller than the maximum difference $\phi_B - \phi_{B'}$, in the basic flow region.

The preceding observations lead to the following assumption which is the fundamental assumption of the proposed method. For a given flow region and for every cut-off of this region, there is a suitable vertical cut XX' of the basic flow region whose image onto plane ω coincides with image ab of the

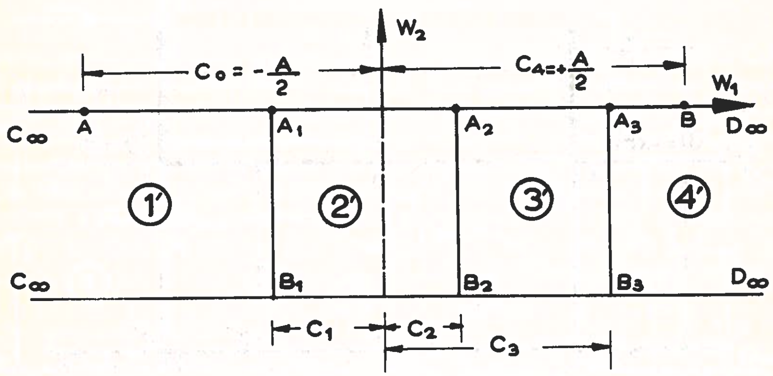

FIG. 3.—IMAGE OF FLOW REGION (1 - α) ONTO PLANE $W = W_1 + i\,W_2$

vertical line through the end of the cut-off. For a given segment XX', x = constant. Putting

$$W(\omega) = W_1(\phi, \Psi) + iW_2(\phi, \Psi) = \ln \frac{1 + m \,\text{sn}\left(\dfrac{2K\omega}{\kappa H} + K\right)}{1 - m \,\text{sn}\left(\dfrac{2K\omega}{\kappa H} + K\right)} \quad \ldots\ldots\ldots (4)$$

along segment XX', $W_1(\phi, \Psi)$ = constant. Thus according to the preceding assumption, the real part of the complex function $W(\omega)$ is constant along each vertical line through the ends of the cut-offs.

As the function $z = (T/\pi)\, W(\omega)$ gives the solution to the flow problem in the basic flow region, the following important properties of this function ensue:

1. Along the upper surfaces of the pervious stratum AC and BD [Fig. 1(a)] the potential $\phi = -\kappa H$ and $\phi = 0$, respectively; consequently $W_2(\phi, \Psi) = 0$.

2. Along the underground contour of the hydraulic structure the streamfunction $\Psi = 0$; consequently $W_2 = 0$.

3. Along the impervious surface CD the streamfunction $\Psi = -Q$; consequently $W_2 = -\pi$.

Because $W_1(\phi, \Psi)$ = constant along each vertical line through the ends of the cut-offs, the orthogonal fragments 1, 2, 3, etc. of the flow region are transformed into the orthogonal fragments 1', 2', 3', etc. on the plane $W = W_1 + iW_2$ (Fig. 3).

It is observed that what was achieved by Pavlovsky's assumption in plane ω is also achieved herein in plane W. The solution procedure is now evident. The Christoffel-Schwarz transformation can be applied to each of the simple fragments of the flow region as well as to the corresponding simple fragments of plane W, to give functions $z(\zeta)$ and $W(\zeta)$. Functions $z(\zeta)$, $W(\zeta)$ and function $W(\omega)$ give the solution to the problem (6).

Now the aforementioned procedure will be applied and a series of equations is derived which gives the solution to the problem.

MATHEMATICAL TRANSFORMATIONS

The three types of fragments in Fig. 4 to be studied are the: upstream fragment [Fig. 4(a)]; intermediate fragments [Fig. (b)] and downstream fragment [Fig. 4(c)].

Upstream Fragment.—The conformal mapping $z - \zeta$ is described by the following Christoffel-Schwarz equation:

$$z = M \int \frac{d\zeta}{\sqrt{\zeta^2 - 1}} + N = -Mi \sin^{-1} \zeta + N \quad \ldots\ldots\ldots\ldots\ldots (5)$$

For point C_1, $\zeta = +1$, $z = 0$ and, therefore, $N = i(M\pi)/2$. For point B_1, $\zeta = -1$, $z = iT$ and, therefore, $M = T/\pi$. Consequently

$$\left.\begin{array}{l} z = -i\dfrac{T}{\pi} \sin^{-1} \zeta + i\dfrac{T}{2} \\ \text{from which } \zeta = \cosh \dfrac{\pi z}{T} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

For point A_1, $\zeta = -a_1$, $z = i(T - S)$ and for point A_0, $\zeta = -a_0$, $z = -L + iT$. Therefore

$$\left.\begin{array}{l} a_0 = \cosh \dfrac{\pi L}{T} \\ a_1 = \cos \dfrac{\pi S}{T} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Now considering conformal mapping $W - \zeta$

$$W = M' \int \frac{d\zeta}{\sqrt{(\zeta - 1)(\zeta + a_1)}} + N'$$

$$= M' \ln \frac{\sqrt{\zeta - 1} + \sqrt{\zeta + a_1}}{\sqrt{\zeta - 1} - \sqrt{\zeta + a_1}} + N' \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

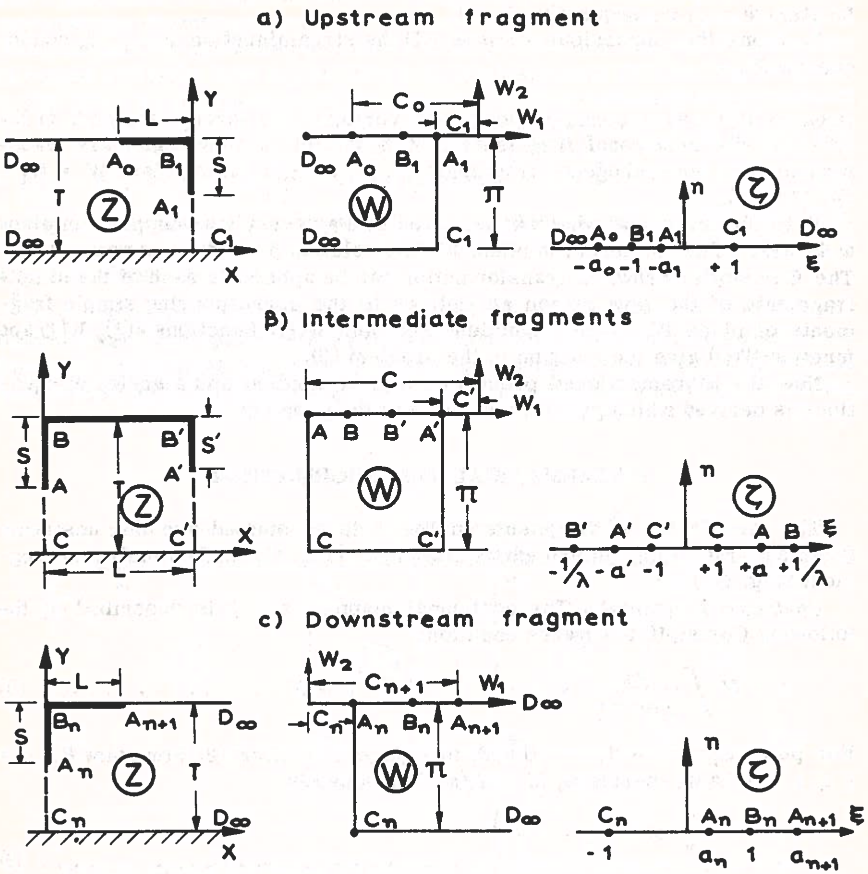

FIG. 4.—FRAGMENTS OF FLOW REGION AND THEIR IMAGES ONTO PLANES W AND ζ

For point A_1, $\zeta = -a_1$, $W = C_1$ and, therefore, $N' = C_1$. For point C_1, $\zeta = +1$, $W = C_1 - i\pi$ and, therefore, $M' = -1$. Consequently

$$W = \ln \frac{\sqrt{\zeta - 1} - \sqrt{\zeta + a_1}}{\sqrt{\zeta - 1} + \sqrt{\zeta + a_1}} + C_1 \quad \text{(9)}$$

For point A_0, $\zeta = a_0$, $W = C_0$ and, therefore

$$C_1 - C_0 = \ln \frac{\sqrt{a_0 + 1} + \sqrt{a_0 - a_1}}{\sqrt{a_0 + 1} - \sqrt{a_0 - a_1}} \quad \text{(10)}$$

Intermediate Fragments.—The conformal mapping $z - \zeta$ is described by the following Christoffel-Schwarz equation:

$$z = M \int \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - \lambda^2 \zeta^2)}} + N = M\,F(\zeta,\lambda) + N \quad \text{(11)}$$

For point C, $z = 0$, $\zeta = 1$, $F(\zeta,\lambda) = K$. For point C', $z = L$, $\zeta = -1$, $F(\zeta,\lambda) = -K$. For point B, $z = iT$, $\zeta = 1/\lambda$, $F(\zeta,\lambda) = K - iK'$. Thus, the following equations are obtained:

$$z = -\frac{L}{2K} F(\zeta,\lambda) + \frac{L}{2} \quad \text{(12)}$$

$$\frac{K}{K'} = \frac{L}{2T} \quad \text{(13)}$$

With the use of tables, Eq. 13 gives parameter λ and the complete elliptic integrals K and K'. Solving Eq. 42 for ζ:

$$\zeta = \mathrm{sn}\left[\left(\frac{K' z}{T} - K\right), \lambda\right] \quad \text{(14)}$$

For point A; $z = i(T - S)$, $\zeta = a$. For point A'; $z = L + i(T - S')$, $\zeta = -a'$. Eq. 14 gives

$$\left.\begin{aligned} a &= \frac{1}{\lambda}\,\mathrm{dn}\left(K' \frac{S}{T}, \lambda'\right) \\ a' &= \frac{1}{\lambda}\,\mathrm{dn}\left(K' \frac{S'}{T}, \lambda'\right) \end{aligned}\right\} \quad \text{(15)}$$

Considering conformal mapping $W - \zeta$:

$$W = M' \int \frac{d\zeta}{\sqrt{(\zeta - a)(\zeta + a')(\zeta - 1)(\zeta + 1)}} + N' \quad \text{(16)}$$

is obtained. Now function u is introduced by means of substitution:

$$\mathrm{sn}\,(u,n) = \sqrt{\frac{(1 + a')(\zeta - a)}{(a + a')(\zeta - 1)}} \qquad n = \sqrt{\frac{2(a + a')}{(1 + a')(1 + a)}} \quad \text{(17)}$$

$$\text{Finally } W = \frac{2M'}{(1 + a)(1 + a')} \int du + N' = \frac{2M'}{(1 + a)(1 + a')} u + N' \quad \text{(18)}$$

is obtained. For point A, $\zeta = a$, sn $u = 0$, $u = 0$ $W = C$. For point A', $\zeta = -a'$, sn $u = 1$ $u = K(n)$, $W = C'$. Therefore

$$W = C + \frac{C' - C}{K(n)} u \quad \text{(19)}$$

For point C', $\zeta = -1$, sn $u = 1/n$, $u = K(n) - iK'(n)$, $W = C' - i\pi$ and, therefore

$$C' - C = \pi \frac{K(n)}{K'(n)} \quad \text{(20)}$$

Downstream Fragment.—The following equations are obtained, corresponding to the upstream strip:

$$\zeta = -\cosh \frac{\pi z}{T} \quad \text{(21)}$$

$$a_n = \cos \frac{\pi S}{T}, \quad a_{n+1} = \cosh \frac{\pi L}{T} \qquad (22)$$

$$W = \ln \frac{\sqrt{\zeta + 1} + \sqrt{\zeta - a_n}}{\sqrt{\zeta + 1} - \sqrt{\zeta - a_n}} + C_n \qquad (23)$$

$$C_{n+1} - C_n = \ln \frac{\sqrt{a_{n+1} + 1} + \sqrt{a_{n+1} - a_n}}{\sqrt{a_{n+1} + 1} - \sqrt{a_{n+1} - a_n}} \qquad (24)$$

At point A_{n+1} (Fig. 4) $\omega = 0$ and, therefore, $\text{sn}\,[(2K\omega)/(\kappa H) + K] = \text{sn}\, K = 1$. Consequently $C_{n+1} = \ln\,[(1 + m)/(1 - m)]$.

At point A_0, $\omega = -\kappa H$ and, therefore, $\text{sn}\,\{[(2K\omega)/(\kappa H)] + K\} = \text{sn}\,(-K) - 1$. Consequently $C_0 = \ln\,[(1 - m)/(1 + m)]$.

Putting $A = \Sigma\,(C_i - C_{i-1})$, $A = C_{n+1} - C_0 = 2 \ln\,[(1 + m)/(1 - m)]$ is obtained. Thus

$$m = \tanh \frac{A}{4} \qquad (25)$$

APPLICATION PROCEDURE

The proposed method is, therefore, applied in the following manner:

1. Eq. 7 is applied and parameters a_0 and a_1 of the upstream fragment are computed. Then Eq. 10 gives difference $C_1 - C_0$.
2. Eq. 13 is applied for each intermediate fragment and parameter λ as well as the complete elliptic integrals $K(\lambda)$, $K'(\lambda)$ are computed. Then Eqs. 15 and 17 give the parameters a, a', n, and the complete elliptic integrals $K(n)$, $K'(n)$ are obtained. Finally, the difference $C' - C$ is computed for each intermediate fragment by means of Eq. 20.
3. Eq. 22 is applied for the downstream fragment and the parameters a_n, a_{n+1} are computed. Then Eq. 24 gives the difference $C_{n+1} - C_n$.
4. The quantities $C_i - C_{i-1}$ are added and the sum $A = C_1 - C_0 + C_{n+1} - C_n + \Sigma\,(C' - C) = C_{n+1} - C_0$ is obtained. It has been shown that $C_{n+1} = -C_0 = A/2$. Therefore, knowing the sum A and the differences $C_i - C_{i-1}$, the quantities $C_0, C_1, \ldots C_{n+1}$ are immediately computed.
5. Eq. $m = \tanh\,(A/4)$ gives the modulus m, so the complete elliptic integrals $K = K(m)$, $K' = K'(m)$ are obtained from tables. Consequently the function $W = \ln \dfrac{1 + m\ \text{sn}\left(\dfrac{2K\omega}{\kappa H} + K\right)}{1 - m\ \text{sn}\left(\dfrac{2K\omega}{\kappa H} + K\right)}$ is absolutely defined. This function results in

$$\omega = -\frac{\kappa H}{2}\left[1 - \frac{1}{K} F\left(\frac{1}{m} \tanh \frac{W}{2},\ m\right)\right] \qquad (26)$$

therefore, parameters $C_0, C_1, \ldots C_{n+1}$ are known and the functions $W(\zeta)$ (Eqs. 9, 19 and 23) are also defined. Consequently, the function (Eq. 26) with functions $W(\zeta)$ and $z(\zeta)$ (Eqs. 6, 14 and 21) gives the solution of the problem, i.e., the complex potential $\omega = \phi + i\Psi$ at every point of the flow region.

The preceding computations are carried out with the use of tables (2,3,5, 7) of elliptic integrals K, K', $F(\zeta, m)$ and of elliptic functions $\text{sn}\,(u, m)$, $\text{dn}\,(u, m)$.

The computation is difficult at interior points of the flow region because of the complex numbers operation. In this case quantities, ϕ, Ψ at the boundary points of the flow region may be computed, then a graphical flow net can easily be drawn, to give ϕ, Ψ at any interior point.

APPLICATION OF PROPOSED METHOD IN DESIGN PROBLEMS

The complete solution of the flow problem is not necessary for design purposes. The designer needs to know seepage discharge Q, pressures p on the underground contour of the hydraulic structures, and exit gradient I_E, i.e., the hydraulic gradient at the toe of the structures. Quantities Q, I_E, and p are computed by means of the proposed method as follows. It is assumed that parameters a_0, a_1, λ, $K(\lambda)$, $K'(\lambda)$, a, a', n, $K(n)$, $K'(n)$, a_n, a_{n+1}, A, C_i, m have been computed in the described manner. The discharge through the given flow region is equal to the discharge through the basic flow region. Therefore

$$Q = \frac{K'(m)}{2K(m)}\,\kappa H \qquad (27)$$

To compute the exit gradient, observe that

$$I_E = \frac{1}{\kappa}\frac{d\phi}{dy} = \frac{1}{\kappa}\frac{d\omega}{dy} = \frac{i}{\kappa}\frac{d\omega}{dW}\frac{dW}{d\zeta}\frac{d\zeta}{dz} \qquad (28)$$

Derivatives $d\omega/(dW)$, $dW/(d\zeta)$ and $d\zeta/(dz)$ are obtained from Eqs. 21, 23 and 26. Then

$$I_E = \frac{\pi H}{4mKT\,\sqrt{(\zeta - 1)(\zeta - a_n)}}\ \frac{\sinh \dfrac{\pi z}{T}}{\text{sn}\,\dfrac{2K\phi}{\kappa H}} \qquad (29)$$

For point B [Fig. 1(*a*)]; $z = iT$, $\phi = 0$, and $\zeta = 1$, therefore, quotient $R = [\sinh\,(\pi Z/T)]/[\text{sn}\,(2K\phi/\kappa H)]$ has the indeterminate form $R = 0/0$. Then for $\zeta \to 1$

$$\lim R = \frac{\left(\sinh \dfrac{\pi z}{T}\right)'}{\left(\text{sn}\,\dfrac{2K\phi}{\kappa H}\right)'} = \frac{\dfrac{\kappa}{T}\cosh \dfrac{\pi z}{T}\dfrac{dz}{d\zeta}}{\dfrac{2K}{\kappa H}\,\text{cn}\,\dfrac{2K\phi}{\kappa H}\,\text{dn}\,\dfrac{2K\phi}{\kappa H}\dfrac{d\phi}{d\zeta}} \qquad (30)$$

Note that $\cosh\,(i\pi) = \cos \pi = -1$, $\text{dn}\,o = 1$, $\text{cn}\,o = 1$ and

$$\lim R = \frac{\pi\kappa H}{2KT}\frac{dz}{d\phi} = -i\frac{\pi H}{2KTI_E} \quad \text{.......................} \quad (31)$$

is obtained. Substituting

$$I_E = \frac{\pi H}{4K(m)T\sqrt{m \sin \frac{\pi S}{2T}}} \quad \text{.......................} \quad (32)$$

is obtained, in which S = the depth of the downstream cut-off and T = depth of the downstream fragment.

Once the magnitude of the exit gradient has been found, the factor of safety with respect of piping is then ascertained by comparing this gradient with the critical gradient $I_{cr} = \gamma'/\gamma$, in which γ' = the submerged unit weight of soil and γ = the unit weight of water.

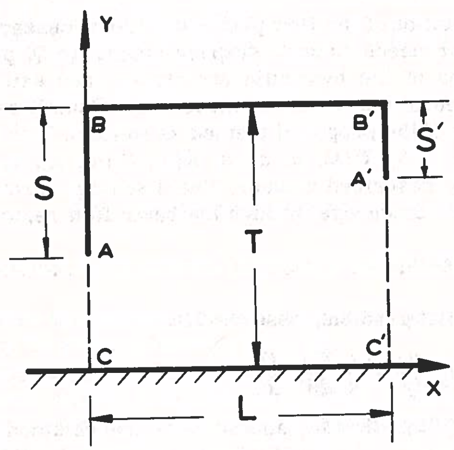

FIG. 5.—INTERMEDIATE FRAGMENT OF FLOW REGION

The pressures on contour ABB'A' are computed as follows (Fig. 5). Pressure p at a given point of ABB'A' is $p = \gamma\ [y' - (\phi/\kappa)]$, in which ϕ = the potential at this point and y' = the vertical distance of the point from the downstream water level. Therefore, potential ϕ at various points of contour ABB'A' needs to be known. According to Eq. 14: $\zeta = \text{sn}\ \{[(K'z/T) - K], \lambda\}$, in which $K/K' = L/(2T)$ and $K = K(\lambda)$, $K' = K'(\lambda)$.

Along segment AB	$z = iy$
Along segment BB'	$z = x + iT$
Along segment B'A'	$z = L + iy$

Putting $\mu = x/L$ and $\rho = (T - y)/T$, yields

$$\left.\begin{array}{l} \text{Along AB} \quad \zeta = \dfrac{1}{\lambda}\,\text{dn}\,(\rho K', \lambda') \\ \text{Along BB'} \quad \zeta = \dfrac{1}{\lambda\ \text{sn}\ [(2\mu + 1)]K, \lambda} \\ \text{Along B'A'} \quad \zeta = -\dfrac{1}{\lambda}\,\text{dn}\,(\rho K', \lambda') \\ \text{in which} \quad \lambda' = \sqrt{1 - \lambda^2} \end{array}\right\} \quad \text{.......................} \quad (33)$$

Once the auxiliary variable ζ is computed at each interesting point of the contour ABB'A', variables u and W are computed by means of Eqs. 17 and 19. Eq. 17 gives for u

$$u = F\left[\sqrt{\frac{(1 + a')(\zeta - a)}{(a' + a)(\zeta - 1)}}, n\right] \quad \text{.......................} \quad (34)$$

Then Eq. 19 gives

$$W = C + \frac{C' - C}{K(n)}\,u \quad \text{.......................} \quad (35)$$

Finally, potential ϕ is computed at each interesting point from Eq. 26, i.e.

$$\phi = -\frac{\kappa H}{2}\left[1 - \frac{1}{K}\,F\left(\frac{1}{m}\tanh\frac{W}{2}, m\right)\right] \quad \text{..................} \quad (36)$$

NUMERICAL VERIFICATION OF PROPOSED METHOD

The accuracy of the proposed method, as well as Pavlovsky's, was checked in the subsequent special flow regions.

In flow region I, four cases were considered as follows:

Case 1	$L/T = 1.00$	$S/T = 0.05$
Case 2	$L/T = 1.00$	$S/T = 0.15$
Case 3	$L/T = 1.00$	$S/T = 0.60$
Case 4	$L/T = 0.50$	$S/T = 0.15$

In flow region II, two cases were considered:

Case 1	$L = 0.9838T$; $T_1 = 0.9998T$; $T_2 = 0.9871T$ $S_1 = 0.5999T$; $S_1' = 0.6001T$; $S_2' = 0.1691T$ $S_2 = 0.1562T$
Case 2	$L = 0.980ST$; $T_1 = T_2 = 1.0146T$ $S_1 = S_2 = 0.3570T$; $S_1' = S_2' = 0.3424T$

In flow region III five cases were considered:

Case 1	$L/T = 0.4230$	$S/T = 0.0273$
Case 2	$L/T = 0.9175$	$S/T = 0.1534$
Case 3	$L/T = 1.1865$	$S/T = 0.3420$
Case 4	$L/T = 1.1290$	$S/T = 0.4364$
Case 5	$L/T = 1.1490$	$S/T = 0.5505$

The proposed method, as well as Pavlovsky's was applied in the aforementioned cases and potential ϕ at characteristic points, exit gradient I_E, and discharge Q were computed. These magnitudes were then compared with the exact ones.

TABLE 1.—DAMS WITH CUT-OFFS (I, II)

Items (1)	Exact values (2)	Proposed Method: Values (3)	Proposed Method: Error, as a percentage (4)	Pavlovsky's Method: Values (5)	Pavlovsky's Method: Error, as a percentage (6)
(*a*) Case I-1					
$Q/\kappa H$	0.519	0.518	0.0	0.543	4.5
$-\phi_B/\kappa H$	0.193	0.192	0.5	0.246	27.5
$-\phi_1/\kappa H$	0.134	0.134	0.0	0.217	62.0
$I_E\ T/H$	1.873	1.868	0.0	2.761	47.0
(*b*) Case I-2					
$Q/\kappa H$	0.488	0.488	0.0	0.496	1.5
$-\phi_B/\kappa H$	0.331	0.332	1.0	0.351	6.0
$-\phi_1/\kappa H$	0.225	0.225	0.0	0.276	22.0
$I_E\ T/H$	1.016	1.016	0.0	1.164	14.5
(*c*) Case I-3					
$Q/\kappa H$	0.339	0.338	0.5	0.338	0.5
$-\phi_B/\kappa H$	0.642	0.641	0.0	0.641	0.0
$-\phi_1/\kappa H$	0.386	0.386	0.0	0.391	1.5
$I_E\ T/H$	0.377	0.377	0.0	0.377	0.0
(*d*) Case I-4					
$Q/\kappa H$	0.649	0.649	0.0	0.658	1.5
$-\phi_B/\kappa H$	0.465	0.465	0.0	0.488	5.0
$-\phi_1/\kappa H$	0.310	0.310	0.0	0.365	18.0
$I_E\ T/H$	1.385	1.382	0.0	1.542	11.0
(*e*) Case II-1					
$Q/\kappa H$	0.322	0.323	0.5	0.327	1.5
$-\phi_1/\kappa H$	0.633	0.631	0.5	0.622	1.5
$-\phi_B/\kappa H$	0.394	0.395	0.5	0.392	0.0
$-\phi_c/\kappa H$	0.218	0.220	1.0	0.235	8.0
$-\phi_2/\kappa H$	0.147	0.152	3.5	0.185	26.0
$I_E\ T/H$	0.651	0.652	0.0	0.741	14.0
(*f*) Case II-2					
$Q/\kappa H$	0.355	0.359	1.0	0.360	1.5
$-\phi_1/\kappa H$	0.732	0.729	0.5	0.710	3.0
$-\phi_B/\kappa H$	0.592	0.590	0.5	0.586	1.0
$-\phi_c/\kappa H$	0.408	0.410	0.5	0.414	1.5
$-\phi_2/\kappa H$	0.268	0.271	0.5	0.290	8.0
$I_E\ T/H$	0.487	0.495	1.5	0.510	4.5

The exact magnitudes can easily be computed in the simple flow region I (4,8), but this is not the case in regions II and III. Thus the following indirect method was applied.

If discharge Q and the values of potential ϕ at the vertices of a flow region are known, the length of the sides of this region can easily be computed. In-

TABLE 2.—EMBEDDED DAMS WITHOUT CUT-OFFS (III)

Items (1)	Exact values (2)	Proposed Method: Values (3)	Proposed Method: Error, as a percentage (4)	Pavlovsky's Method: Values (5)	Pavlovsky's Method: Error, as a percentage (6)
(*a*) Case III-1					
$Q/\kappa H$	0.726	0.749	3	0.886	22
$J_o\ T/\kappa H$	1.340	1.352	1	0.913	32
$-\phi_c/\kappa H$	0.125	0.150	20	0.307	145
$I_E\ T/H$	3.070	3.850	25	7.160	133
(*b*) Case III-2					
$Q/\kappa H$	0.425	0.438	3	0.454	7
$J_o\ T/\kappa H$	0.582	0.583	0	0.536	8
$-\phi_c/\kappa H$	0.160	0.204	27	0.254	59
$I_E\ T/H$	0.805	0.903	12	1.049	30
(*c*) Case III-3					
$Q/\kappa H$	0.286	0.294	3	0.295	3
$J_o\ T/\kappa H$	0.450	0.454	1	0.446	1
$-\phi_c/\kappa H$	0.177	0.218	23	0.235	33
$I_E\ T/H$	0.381	0.410	8	0.426	12
(*d*) Case III-4					
$Q/\kappa H$	0.254	0.257	1	0.261	3
$J_o\ T/\kappa H$	0.460	0.460	0	0.465	1
$-\phi_c/\kappa H$	0.189	0.232	23	0.238	26
$I_E\ T/H$	0.308	0.323	5	0.333	8
(*e*) Case III-5					
$Q/\kappa H$	0.209	0.214	2	0.213	2
$J_o\ T/\kappa H$	0.478	0.478	0	0.473	1
$-\phi_c/\kappa H$	0.194	0.226	16	0.228	17
$I_E\ T/H$	0.234	0.244	4	0.244	4

deed, the Christoffel-Schwarz equation 1 gives

$$\omega = \frac{\kappa H}{2K} F(\zeta, m) - \frac{\kappa H}{2} \quad \text{. . . . . (37)}$$

$$\zeta = \text{sn}\left[\left(\frac{2K\omega}{\kappa H} + K\right), m\right] \quad \text{. . . . . (38)}$$

in which $\frac{K'(m)}{K(m)} = \frac{2Q}{\kappa H}$ . . . . (39)

For given Q, the complete elliptic integrals K, K' and the modulus m from Eq. 39 may be obtained. Consequently, knowing the magnitude $\omega = \phi$ at each vertex of the flow region, Eq. 38 can be applied to give the values of auxiliary variable ζ, i.e., parameters a_i, c_i of the Christoffel-Schwarz integral 2. Thus, this integral becomes a definite integral which can be computed with the desirable approximation to give the dimensions of the flow region. In cases II and III this integral is expressed by means of elliptic integrals of the first, second and third kind (8). Thus, computations are facilitated.

If parameters m, a_i, c_i, are known, the exit gradient I_E can easily be computed. The Christoffel-Schwarz equation 2 can also be written as

$$\left.\begin{aligned} dz &= M \frac{d\zeta}{(\zeta^2 - \gamma^2)\sqrt{\zeta^2 - 1}} \sigma(\zeta) \\ \text{in which } \gamma &= \frac{1}{m}, \quad \sigma(\zeta) = \frac{\Pi(\zeta - a_i)}{\Pi\sqrt{\zeta - c_i}} \end{aligned}\right\} \qquad (40)$$

If δ is a very small positive quantity, it can be assumed that function $\sigma(\zeta)/\sqrt{\zeta^2 - 1}$ = constant = $\sigma(\gamma)/\sqrt{\gamma^2 - 1}$ when $\gamma - \delta \leqq \zeta \leqq \gamma + \delta$. Consequently

$$\Delta z = \frac{M\sigma(\gamma)}{\sqrt{\gamma^2 - 1}} \int_{\gamma-\delta}^{\gamma+\delta} \frac{d\zeta}{\zeta^2 - \gamma^2} = - \frac{M\sigma(\gamma)}{2\gamma\sqrt{\gamma^2 - 1}} \ln \frac{\delta + 2\gamma}{\delta - 2\gamma} \qquad (41)$$

is obtained. For $\delta \rightarrow 0$ lim ln $[(\delta + 2\gamma)/(\delta - 2\gamma)] = \ln(-1) = i\pi$ and lim $\Delta z = - iT_2$ [Fig. 1(a)]. Thus

$$M = \frac{2T_2\gamma\sqrt{\gamma^2 - 1}}{\pi\,\sigma(\gamma)} = \frac{2T_1\gamma\sqrt{\gamma^2 - 1}}{\pi\,\sigma(-\gamma)} \qquad (42)$$

is obtained, and

$$I_E = \frac{1}{\kappa}\frac{d\phi}{dS} = \frac{i}{\kappa}\frac{d\omega}{dz} \qquad (43)$$

From Eq. 37 $$d\omega = \frac{\kappa H\gamma}{2K} \frac{d\zeta}{\sqrt{(\zeta^2 - \gamma^2)(\zeta^2 - 1)}} \qquad (44)$$

From Eqs. 40 and 42 $$dz = \frac{2T_2\gamma\sqrt{\gamma^2 - 1}\,\sigma(\zeta)}{\pi\sigma(\gamma)(\zeta^2 - \gamma^2)\sqrt{\zeta^2 - 1}} \qquad (45)$$

Consequently $$\frac{d\omega}{dz} = \frac{\kappa\,\pi\,H\,\sigma(\gamma)}{4KT_2\sqrt{\gamma^2 - 1}} \frac{\sqrt{\zeta^2 - \gamma^2}}{\sigma(\zeta)} \qquad (46)$$

At point B [Fig. 1(a)] $\zeta = 1$. Therefore

$$I_E = \frac{\pi\,H\,\sigma(\gamma)}{4K\,T_2\sigma(1)} \qquad (47)$$

This was the method followed herein. Flow regions II and III were constructed to have definite values Q and ϕ at their vertices. The exit gradient was computed from Eq. 47. Then, the proposed method, as well as Pavlovsky's was applied and the derived Q, ϕ, I_E values were compared with the a priori known exact ones. The results of computations are given in Tables 1 and 2.

It is observed that the results of the proposed method almost coincide with the exact ones, in flow region I and II, i.e., in cases of a flat bottom dam with cut-offs at the one or both ends. In flow region III the observed errors are considerable because of the special characteristics of this region.

In the vicinity of B, C (Fig. 6), slope angles a of the stream lines are great, while the proposed method assumes that $a = 0$. Therefore, the images of a small upper part of BB', CC' on the plane ω are significantly different from the images of the corresponding upper part of XX' (Fig. 2). If the embedded structure III has two small cut-offs at B and C, the accuracy of the proposed method is considerably improved. To verify this, the subsequent special flow region was considered. The cut-off depth S' is only one-tenth of the foundation depth S (Fig. 7).

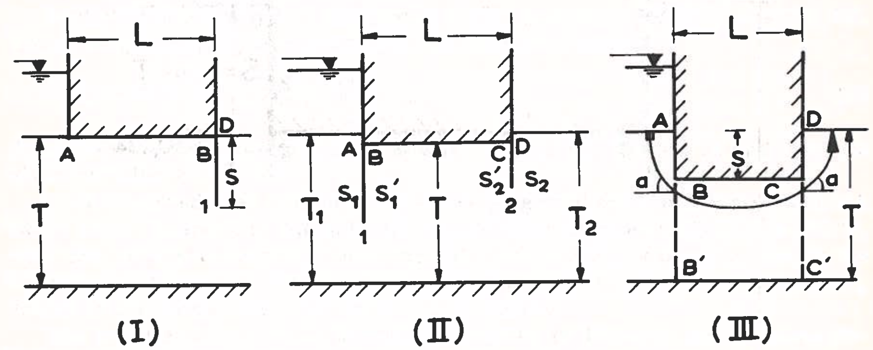

FIG. 6.—SCHEMES USED FOR NUMERICAL VERIFICATION

The following results were derived:

Accurate Results

$$\phi_c = -0.256\,\kappa H, \quad \phi_2 = -0.232\,\kappa H, \quad I_E = 0.390\frac{T}{H}, \quad Q = 0.308\,\kappa H$$

Results of Proposed Method.—The results of the proposed method are given as follows:

$\phi c = -0.277\,\kappa H$	Error 8 %
$\phi_2 = -0.257\,\kappa H$	Error 10 %
$I_E = 0.412\,T/H$	Error 6 %
$Q = 0.314\,\kappa H$	Error 2 %

It is observed that the error of ϕ_2 is considerably smaller than the error of ϕ_c (23 %) in the corresponding flow region III-3. The error becomes smaller as the quotient S'/S becomes greater.

The relative inaccuracy of the proposed method in case III seems to be restricted at the vicinity of the singular points B and C.

In the case under consideration, Eqs. 44 and 45 give

$$J_x = \frac{d\phi}{dx} = \frac{\pi\kappa H}{4KT}\sqrt{\frac{(\gamma^2 - \alpha^2)(\gamma^2 - \xi^2)}{(\gamma^2 - 1)(\alpha^2 - \xi^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48)$$

in which J_x = the potential gradient along BC = v_x and

$$\alpha = \xi_C = - \xi_B = \text{sn}\left[\left(\frac{2K\phi_C}{\kappa H} + K\right), m\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (49)$$

The gradient J_x is nearly constant along BC, except at the vicinity of points B and C where it increases quickly and then becomes infinite at these points. Consequently the potential distribution is as shown in Fig. 8. This distribu-

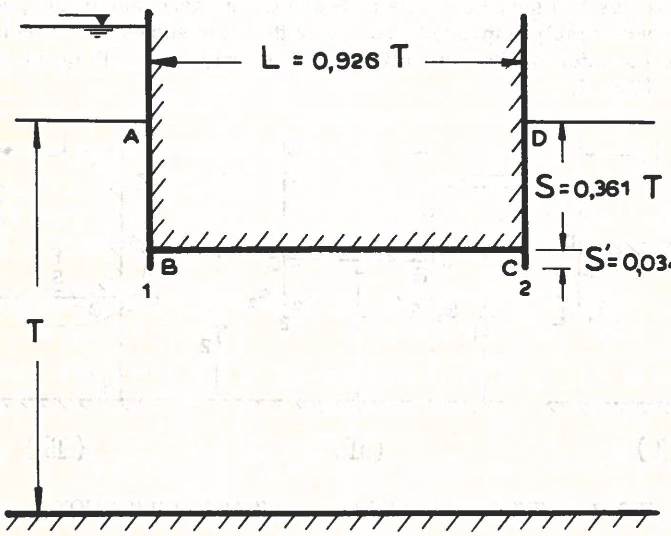

FIG. 7.—FLOW REGION WITH SMALL CUT-OFFS

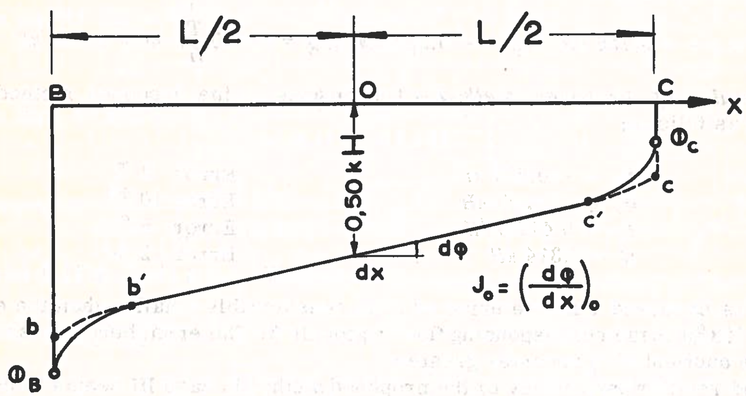

FIG. 8.—DISTRIBUTION OF POTENTIAL ALONG BASE OF EMBEDDED DAM

tion is, basically, defined by the gradient J_o at the middle point 0. From Eq. 48 the following equation is obtained for $\xi = 0$.

$$J_o = \frac{\pi\kappa H\gamma}{4KT\alpha}\sqrt{\frac{\gamma^2 - \alpha^2}{\gamma^2 - 1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (50)$$

On the other hand, the proposed method gives

$$J_x = \frac{\pi\kappa H}{4K_0 m_0 \ \text{sn}\left(\frac{2K_0\phi}{\kappa H}\right)(T - S)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (51)$$

in which (Eq. 25) m_0 = tanh $(A/4)$. At point O $\phi = -\kappa H/2$. Consequently

$$J_o = \frac{\pi\kappa H}{4K_0 m_0 (T - S)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

Given $0 < |\phi_B|, |\phi_C| < \kappa H$, gradient J_x does not become infinite at points B, C, according to the proposed method.

Pavlovsky's assumption involves a linear potential distribution along BC. Thus, the gradient J_o is according to Pavlovsky's method:

$$J_o = \frac{\phi_C - \phi_B}{L} = \frac{\kappa H + 2\phi_C}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

Quantities J_o were computed by means of Eqs. 50, 52 and 53 and they are given in Table 2.

It is observed that gradients J_o of the proposed method coincide with the exact ones. Therefore, the potential distribution is given in Fig. 8 by bb'c'c, i.e., the errors concerning the computation of potential are restricted in the vicinity of points B, C. Pavlovsky's method also gives accurate values for J_o except in the cases where the ratio S/T is small.

The results of the whole checking are given in Tables 1 and 2. These results are summed as follows:

1. In all the examined cases of a flat bottom dam with cut-offs at one or both ends, the results of the proposed method almost coincide with the exact ones.
2. In all the examined cases of an embedded dam without cut-offs, the computed ϕ_C values at the end C are considerably different from the corresponding exact ones. Considerable errors are also observed for exit gradient I_E, when exit point D is near the singular point C. Except in the vicinity of the end points B and C, the computation of ϕ values at intermediate points of the base BC seems to be accurate. The computation of discharge Q is also very satisfactory.
3. In the examined case of an embedded dam with two very small cut-offs at the ends B and C, the errors of the computation in the vicinity of these points are considerably smaller than the corresponding errors in case of the analogous embedded dam without cut-offs. It seems therefore that the proposed method is satisfactory in case of a flat bottom or embedded dam with cut-offs at the ends.
4. The results of Pavlovsky's method are less accurate than the results of the proposed method in all the cases examined. Pavlovsky's method gave satisfactory results in cases where the ratio S/T was large, but inaccurate results were obtained in cases where the ratio S/T was small.

It is easy to verify that in the cases characterized by small S/T values, Chugaev's simple method gives more accurate results than Pavlovsky's method, except for discharge Q, for which the results of Chugaev's method are less accurate. It is also easy to verify that in many of the examined cases III, Chugaev's method gives more accurate ϕ_C and I_E values than the proposed method. But Chugaev's method assumes that the potential ϕ is linearly distributed along the base BC of the dam; therefore the erroneous straight line from ϕ_B to ϕ_C is finally obtained (Fig. 8). It must also be noted that in several cases III, Chugaev's method considerably underestimates the discharge.

CONCLUSIONS

The present work introduces a new approximate analytical solution of the confined flow problem under hydraulic structures. The proposed method is an improvement of Pavlovsky's method of fragments.

The method is applicable under the following basic assumptions.

1. The pervious stratum can be considered as homogeneous and isotropic and Darcy's law is valid.
2. The flow can be considered as two dimensional.
3. The contour of the flow region is composed of horizontal and vertical straight line segments.

The proposed method can give the complete solution of the flow problem, i.e., the ϕ, Ψ values at any point of the flow region.

The method is based on a clear assumption. It was assumed that the images in plane $\omega = \phi + i\Psi$ of the vertical lines dividing the flow region into fragments are similar to the images of proper vertical lines through the flow region of the flat bottom dam without cut-offs. Afterwards a rigorous mathematic treatment was applied to give the solution to the problem.

The fundamental assumption of the proposed method is satisfactory in case the aforementioned vertical dividing lines are extensions of cut-offs and the numerical results almost coincide with the exact ones.

In case the upper point of a dividing vertical line is a vertex of the foundation contour at which the interior angle is $3\pi/2$, the fundamental assumption of the proposed method is no longer satisfactory in the vicinity of the vertex. Therefore inaccurate results are obtained in this vicinity. Anyway the method seems to give satisfactory results concerning discharge and ϕ values except in the vicinity of the vertices where, on the other hand, the real flow pattern is not clear.

The proposed method involves more computation work in comparison to the other approximate methods, especially Chugaev's, but the computation procedure is systematic and it does not require mathematic investigations. Consequently the aforementioned disadvantage of the method may be considered of little importance.

APPENDIX I.—REFERENCES

1. Browzin, B. S., *Nouvelle méthode d'application de quelques fonctions de la variable complexe aux calcules de sous-pressions agissant sous les ouvrages* "La Houille Blanche," France, November, 1964.
2. Dwight, H., *Mathematical Tables*, Dover Publications, Inc., New York, 1961.
3. Hancock, H., *Elliptic Integrals*, Dover Publications, Inc., New York, 1958.
4. Harr, M. E., *Groundwater and Seepage*, McGraw-Hill Co., Inc. New York, 1962.
5. Jahnke-Emde-Lösch, *Tables of Higher Functions*, McGraw-Hill Co., Inc., New York, 1960.
6. Leliavsky, S., *Irrigation and Hydraulic Design* Vol. 1, Chapman and Hall, London, England, 1955.
7. Milne-Thomson, *Jacobian Elliptic Functions-Tables*, Dover Publications Inc., New York, 1950.
8. Polubarinova-Kochina *Theory of Groundwater Movement*, translated by J. M. Roger de Wiest, Princeton Univ. Press, Princeton, N.J., 1962.
9. Tsertoousov, M. D., *Hydraulics*, Moskow, U.S.S.R., 1962, (in Russian).

APPENDIX II.—NOTATION

The following symbols are used in this paper:

A, C_i = constants;
cosh u = hyperbolic cosine of u;
cn (u,m) = elliptic cosine of u for modulus m, i.e., cn (u,m) = $\sqrt{1 - \text{sn}^2 (u,m)}$;
dn (u,m) = $\sqrt{1 - m^2 \text{sn}^2 (u,m)}$;
$F(\zeta,m)$ = elliptic integral of first kind for modulus m, i.e., $F(\zeta,m)$ = $\int_0^\zeta [d\zeta/\sqrt{(1 - \zeta^2)(1 - m^2\zeta^2)}]$;
H = total hydraulic head [see Fig. 1(*a*)];
I_E = exit gradient;
J = potential gradient;
$K(m)$ = $F(1,m)$ = complete elliptic integral of first kind;
$K'(m)$ = $F(1,m')$ = complete elliptic integral of first kind;
L = length (horizontal);
ln u = Naperian logarithm of u;
M, M' = constants;
m,n = moduli of elliptic integrals and functions;
$m'n'$ = comoduli of elliptic integrals and functions, i.e., $m' = \sqrt{1 - m^2}$ and $n' = \sqrt{1 - n^2}$;
N, N' = constants of integration;
p = pressure;
Q = discharge (per unit normal to direction of flow);
S = depth of cut-offs, depth of foundation;
sinh u = hyperbolic sine of u;
sn (u,m) = elliptic sine of u, for modulus m, i.e., if $u = F(\zeta,m)$, sn (u,m) = ζ;
T = thickness of pervious stratum;
tanh = hyperbolic tangent of u;
v = seepage velocity;
W = $W_1 + W_2$ = auxiliary complex function (see Eq. 4);

$z = x + iy$ = complex coordinate of point in flow region;
$\gamma = 1/m$ = unit weight;
Δ = difference [see Fig. 1(*b*)];
$\zeta = \xi + i\eta$ = point of upper half plane;
κ = coefficient of permeability;
λ = modulus of elliptic integrals and functions;
$\lambda' = \sqrt{1 - \lambda^2}$ = comodulus;
Π = product (see Eq. 2);
$\sigma(\zeta)$ = function of ζ as defined in text (see Eq. 40);
ϕ = potential function (velocity potential);
Ψ = stream function; and
ω = complex potential.

Journal of the
SOIL MECHANICS AND FOUNDATIONS DIVISION
Proceedings of the American Society of Civil Engineers

DISCUSSION

Engineering Properties of Mine Tailings[a]

Closure by Howard C. Pettibone[4], M. ASCE and C. Dan Kealy[5]

The writers thank Jackson for his comments on several possible pollution hazards. His discussion fills an acknowledged void in the paper and is a welcome supplement. The discusser's point of caution is well made and we agree that good engineering judgment should be equally applied to the structural and pollution aspects of any proposed tailings usage.

Approximate Solution to Flow Problems Under Dams[b]

Discussion by Boris S. Browzin,[2] F. ASCE and Larry A. White,[3] M. ASCE

The analysis of ground-water flow under hydraulic structures, e.g., dams primarily, has considerable practical importance during the design of these structures when situated on pervious soils. At present the tedious procedure of plotting the so-called flow net is used. It is true that the flow net as a graphical procedure is a solution to the problem, but the underground configuration of a dam, which necessarily must have more than one cut-off, makes this procedure graphically difficult. It is said that experience is essential when plotting the flow net, but even in specialized design offices the flow net is not considered a routine job, consequently there are no experienced flow net plotters available for relatively quick and accurate flow net construction. Moreover, several alternative proposals usually exist before a final one is selected and for each proposal a flow net is needed. Consequently, the amount of work required for the study of uplift based on the flow net requires considerable time.

For this reason, efforts of researchers to recommend an exact or sufficiently accurate and rapid method utilizing analytical methods is important. Christoulas

[a]September, 1971, by Howard C. Pettibone and C. Dan Kealy (Proc. Paper 8382).

[4]Research Civ. Engr., Spokane Mining Research Center, U.S. Bureau of Mines, Spokane, Wash.

[5]Mining Engr., Spokane Mining Research Center, U.S. Bureau of Mines, Spokane, Wash.

[b]November, 1971, by Demetrius G. Christoulas (Proc. Paper 8528).

[2]Chf., Soil and Civ. Engrg. Branch, Bureau of Design and Engrg., Government of the Dist. of Columbia, Washington, D.C.

[3]Civ. Engr., U.S. Army Corps of Engrs., Baltimore Dist.

is to be commended for his work and use of the extensive Russian literature on ground-water flow, to present a working method for determining uplift. Browzin presented another method in a publication in French (10), for the same purpose.

Practicing engineers should not be afraid of using analytical methods if the methods are presented in a convenient manner. The improvement of Pavlovskii's method presented by Christoulas belongs to this type of publication, i.e., it offers a practical tool for the engineer. However, a numerical example of the method is considered necessary; one was not provided in the paper.

The well-known Schwarz-Christoffel transformation, that maps the half space including the x-axis into a polygon, is called by Christoulas, following the Soviet innovation, the Christoffel-Schwarz transformation, as opposed to the tradition in western literature. This innovation is based on the fact that Christoffel (1829–1900) published his work in 1867, 2 yr earlier than Schwarz (1843–1921), i.e., in 1869. However, Schwarz treated it as a mathematical problem, whereas Christoffel as a problem of application, to solve a heat problem, each discovering the method independently.

The transformation is provided by the integral

$$z = A \int_0^{\zeta} (\zeta - a_1)^{\alpha_1 - 1} (\zeta - a_2)^{\alpha_2 - 1} \dots (\zeta - a_n)^{\alpha_n - 1} \, d\zeta + B \quad \dots\dots\dots (54)$$

in which $\xi\eta$, ($\zeta = \xi + i\eta$), designate the positive half plane and xy ($z = x + iy$) the plane in which the polygon is located. The terms $\alpha_n\pi$ represents the magnitude of the polygon angles; a_n = the abscissa of the apexes in the ζ-plane; and A and B are complex constants. Eq. 54 is called the Schwarz-Christoffel transformation or integral. From Eq. 54 the two equations in Christoulas' paper may be derived among many others, each particular equation is usually called a mapping function. It would certainly be of interest to the reader if the author could present the derivation of the two mapping functions (Eqs. 1 and 2) from Eq. 54 or indicate a literature source of the derivation, because Eqs. 1 and 2 represent the basis of the method. It would be interesting to compare the approach of earlier authors to that of Christoulas.

The concept of the Pavlovskii's approximate method consists of subdividing the field of flow (under dams) into fragments. From the lower end points of the cutoffs, Pavlovskii traces verticals downwards, dividing the field of flow, as was mentioned by Christoulas. The method results in relatively large errors, but Pavlovskii indicated a way to obtain further improvement. This was done by Chertousov (equivalent to Tsertoousov in Christoulas paper, Ref. 9) and independently by Chugaev (11,12). Chertousov divides the field of flow by an approximate tracing of equipotential lines from the intersection of the cutoffs with the line of contact of the dam bottom with the soil. The equipotential lines begin as a tangent to the bisectrices and end as a tangent to the verticals to the impervious boundary below the soil stratum. The accuracy of tracing these equipotentials have little influence on the result. Chugaev proposed a method based on approximate determination of the coefficients of hydraulic resistance for each portion (fragment) of the field of flow by adaptation of theoretical solutions of flow for conditions identical or similar to flow conditions in each fragment. The method was included in 1957 in Code Requirements of the Ministry of Electric Power Stations of his country for the design of dams and made the subject of a book (12). Chertousov's method is not mentioned by the author and reference to the original Chugaev publication (11,12) is not provided.

These two improvements of Pavlovskii's method, firstly, Ref. 9 and secondly, Refs. 11 and 12, provide a complete solution of the problem for engineering purposes, i.e., the determination of the uplift, the flow gradients, and velocity, including the danger of piping at the toe and the ground-water discharge. However, the merit of Christoulas' work consists in providing further research in the analytical development of Pavlovskii's method and presenting calculations proving the accuracy of this method. Nevertheless, the writers do not think that an improvement of Pavlovskii's method is the correct way to offer to the engineering profession a substitute for the old-fashioned flow net procedure. The assumption on which the author's method is based is this. The fragment 1 downstream border is represented by the line $A_1 B_1$ [Fig. 1(*a*)]. Its image in complex potential plane ω is curve $a_1 b_1$. Because of the conformal representation, α [Fig. 1(*a*)] and α′ [Fig. 1(*b*)] are of the same magnitude at point M and in its image M′. However, it is incorrect (perhaps it is a typographic error) to say that "this angle (i.e., α) vanishes at point A and at point B. Angle α is $\pi/2$ at A and at B because the streamline is vertical at these points. But angle α vanishes at points A_1 and B_2 because at these points the streamline is horizontal. The author further considers a simple region called a basic flow region in which a vertical line XX′ exists, the image of which in plane ω [Fig. 1(*b*)] is also the line $a_1 b_1$, i.e., the line $a_1 b_1$ is the image which is common to both, the line $A_1 B_1$ [Fig. 1(*a*)] and the line XX′ (Fig. 2). Then the mathematical descriptions of the line XX′ are found by the author and consequently the line $a_1 b_1$. This procedure is certainly very artificial, but since it provides the solution, the method may be developed into a working tool.

The writers, however, believe that the right way to develop the theory, and to find an analytical method for the calculation of the uplift on dams and for other ground flow characteristics, consists of using the method of successive conformal transformations of the flow region of arbitrary configurations until the region becomes a simple case of a flat dam without cutoffs. To illustrate this method consider flow region z [Fig. 9(*b*)] and its image in the positive half plane ζ [Fig. 9(*a*)]. The Schwarz-Christoffel integral, Eq. 54, provides mapping function z to transform plane ζ into z. The reverse transformation from z to ζ may be obtained only for few problems but not in general. The angles of polygons 2, 3, 4, and 6 at infinity representing the space below the dam are $\pi/2$, 2π, $\pi/2$, and 0, respectively, therefore, $\alpha_2 = 1/2$, $\alpha_3 = 2$, $\alpha_4 = 1/2$, and $\alpha_6 = 0$. Taking into account that in the integral of Eq. 54 it is possible to introduce arbitrarily three constants for the values of a_1, a_2, ... a_n, the following values will be assigned: $a_2 = -s$; $a_3 = 0$; and $a_4 = S$; in which S is the length of the cutoff. Introducing these values into Eq. 54

$$Z = A \int_0^{\zeta} \frac{\zeta\, d\zeta}{\sqrt{\zeta^2 - S^2}} + B \quad \text{(55)}$$

and $$Z = A\sqrt{\zeta^2 + S^2} + B \quad \text{(56)}$$

Constant B is determined from known values at point 4, $Z = 0$, and $\zeta = S$; then $B = 0$. At point 3, $Z = iS$ and $\zeta = 0$, then $A = 1$, so

$$Z = \sqrt{\zeta^2 - S^2} \quad \text{(57)}$$

and consequently $$\zeta = \sqrt{Z^2 + S^2} \quad \text{(58)}$$

This is the mapping function transforming plane Z into plane ζ and developing the cutoff into straight line 2-3-4 in plane ζ. The dam underground contour transformed into the straight line makes possible the application of the solution for a dam without cutoff to the dam with one cutoff. The remarkably simple mapping function, Eq. 58, makes even possible to use the graphical procedure (11) for obtaining the uplift values at any point fom 1 to 5 [Fig. 9(*b*)]. Unsymmetrical cutoff requires an additional conformal shift.

If the dam has a second cutoff, the mapping of this second cutoff into plane ζ distorts it into a slightly curved line. The same mapping function, Eq. 58, is then applied a second time, to develop the second cutoff into a horizontal line, i.e., plane ζ is transformed into plane ζ_1 and if necessary a third successive

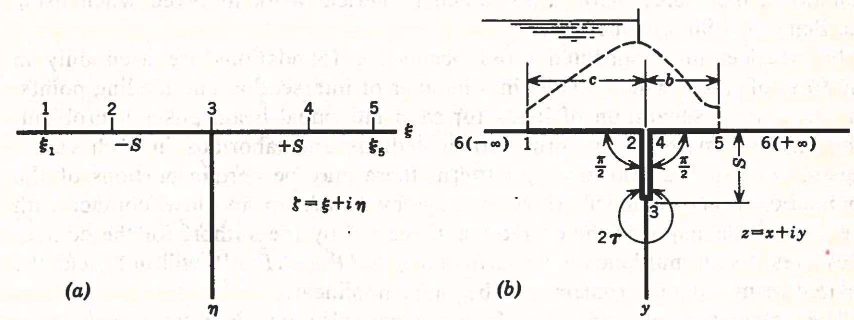

Fig. 9.—Transformation of Half-Plan ζ into Polygon Containing Cut-Off S in Plan Z

transformation is used to map plane ζ_1 into plane ζ_2. This process finally transforms any arbitrary underground contour into a straight line. Because of the simplicity of the mapping function, Eq. 58, the practical application of the method of successive conformal mapping in engineering design may become a routine procedure when the method is further investigated and offered to the profession in a comprehensive presentation.

APPENDIX.—REFERENCES

10. Browzin, B. S., "Nouvelle Méthode d'Application de Quelques Fonctions de la Variable Complexe aux Calcules de Sous-Pressions Agissant Sous les Ouvrages de Retenue (A new method for the application of some complex variable fonctions to the calculation of the uplift action on dams), *La Houille Blanche*, No. 7, France, 1964.
11. Chugaev, R. R., "Proektirovanie podzemnogo kontura plotin raspolozhennykh na neskal-nykh gruntakh," *Izvestiia* VNIIG, 1955, Vol. 53, 1956, Vol. 56.
12. Chugaev, R. R., *Podzemnyi kontur gidrotekhni Cheskikh Sooruzhenii, Gosenergoizdat*, 1962.

PREPARING AND SUBMITTING DISCUSSIONS

Discussion of a Proceedings Paper is open to anyone who has significant comments or questions regarding the technical content of the paper. Discussions are accepted for a period of four full months following the date of publication of a paper. They should be sent to the Editor of Technical Publications, ASCE, 345 East 47 Street, New York, New York 10017. The discussion period may be extended by a written request from a discusser.

The original and two copies of the Discussion should be submitted on 8-1/2-in. by 11-in. *white bond paper* (not tissue), typed double-spaced with wide margins. The length of a Discussion is restricted to two Journal pages (about four typewritten double-spaced pages of manuscript *including* illustrations); the editors will delete matter extraneous to the subject under discussion. If a Discussion is over 2 pages long it will be returned for shortening. All Discussions will be reviewed by the editors and/or the Division's Publications Committee. In some cases, Discussions will be returned to discussers for rewriting, or they may be encouraged to submit a paper or Technical Note rather than a Discussion.

Standards for Discussions are the same as for Proceedings Papers. A Discussion is subject to rejection if it contains matter readily found elsewhere, advocates special interests, is carelessly prepared, controverts established fact, is purely speculative, introduces personalities, or is foreign to the purposes of the Society. All Discussions should be written in the third person, and the discusser should use the term "the writer" when referring to himself. The author of the original paper is referred to as "the author."

Discussions have a specific format. The title of the original paper appears at the top of the first page with a superscript which corresponds to a footnote indicating the month, year, author(s), and number of the original Paper. The discusser's full name should be indicated below the title, and again to introduce the first paragraph (see Discussions herein as an example), together with his ASCE membership grade (if applicable).

The discusser's title, company affiliation, and business address should appear on the first page of the manuscript, along with the number of the original paper, the date and name of the Journal in which it appeared, and the author's name.

Note that the discusser's identification footnote should follow consecutively from the original paper. If the paper under discussion contained footnote numbers 1 and 2, the first Discussion would begin with footnote 3, and subsequent discussions would continue in sequence.

Figures supplied by the discusser should be designated by letters, starting with A. This also applies separately to tables and references. In referring to a figure, table, or reference that appeared in the original paper use the number used by the author.

It is suggested that potential discussers request a copy of the ASCE Author's Guide for more detailed information on preparation and submission of manuscripts.

Approximate Solution to Flow Problem under Dams [a]

Closure by Demetrius G. Christoulas,[4] M. ASCE

Browzin and White's interest in the writer's paper is appreciated very much. Browzin and White consider the writer's method as a working one, but they also consider it as very artificial. They believe that the right way to find an analytical solution to flow problem under hydraulic structures consists of using the method of successive transformations presented by Browzin (1,10) and mentioned by the writer in his paper (1).

The writer believes that an approximate method must be valued on the basis of the following characteristics: (1) Clarity of the assumptions on which the approximate method is based so that the application field of the method is discernible; (2) accuracy and applicability of the method in a broad field; (3) completeness of the method; and (4) computational work required in the application of the method.

The writer's method is based on a clear and reasonable assumption. This assumption is very satisfactory in a broad application field (deep and short cut-offs). The method is also complete since it can give the Φ, φ values in any point of the flow region. As the writer stated in his paper (Conclusions) his method involves more computational work in comparison to the other approximate methods, especially Chugaev's, but the computation procedure is systematic and it does not require mathematic investigations. Consequently, the aforementioned disadvantage of the method may be considered of little importance.

Browzin and White, in their discussion, gave an idea of the method of successive transformations presented in detail by Browzin in 1964 (1,10). This method is especially suitable in case of a pervious stratum of infinite depth. For a stratum of finite depth T the transformation $\zeta = \sqrt{z^2 + s^2}$ gives a curved impervious boundary $C_\infty D_\infty$ [Fig. 1(a)] in plane ζ, approximated by a mean straight line. In case the S/T values are small the curving of the image of $C_\infty D_\infty$ is light, so the latter approximation does not introduce considerable error. In case the S/T values are great, the approximation by a mean straight line may introduce considerable error, so Browzin considered (1,10) the method as applicable only in case $S/T \leqq 0.50$.

It may be also noted that in case of finite depth T the method of successive transformations requires, as the writer's method does, the use of elliptic integrals of 1st kind.

It is true, that the model of homogeneous pervious stratum is very often far from reality. On the other hand, the problem in case of heterogeneity can be solved through numerical methods. However, the writer believes that the search for more accurate analytical methods on the basis of homogeneity is justified. The analytical methods are inexpensive, easily applicable, and they

[a] November, 1971, by Demetrius G. Christoulas (Proc. Paper 8528).
[4] Hydr. Consulting Engr., Athens, Greece.

give a clear insight to the problem. Consequently, they are preferable in case the model of homogeneous pervious stratum is considered as satisfactory or the lack of sufficient field measurements makes it necessarily acceptable. The writer believes that, in these cases, the application of the more accurate analytical methods is justified since the addition of computational errors to the modeling errors is so avoided without excessive work in comparison to less accurate methods. It must also be noted that some problems of determining the flow characteristics for structures founded in an heterogeneous pervious stratum can be reduced to one for a homogeneous pervious stratum. Polubarinova-Kochina has suggested such a method in case the pervious stratum is composed of two layers of different permeability (4,8).

A question raised in the discussion concerns the derivation of mapping functions 1 and 2. The functions, 1 and 2, immediately result from Christoffel-Schwarz transformation (Eq. 54) on the basis of Figs. 1(*a*), 1(*b*), 1(*c*). For the rectangle ω [Fig. 1(*b*)] $\alpha_1\pi = \alpha_2\pi = \alpha_3\pi = \alpha_4\pi$, $a_1 = -1/m$, $a_2 = -1$, $a_3 = -1$, $a_4 = +1/m$. Consequently, function 1 is obtained. For the polygon, z, the angles, $\alpha\pi$, take the following values: (1) $\alpha\pi = 0$ at the vertices $C_\infty(a_1 = -1/m)$, $D_\infty(a_n = -1/m)$; (2) $\alpha\pi = 1/2$ at the vertices A $(a_2 = -1)$, B $(a_{n-1} = +1)$; (3) $\alpha\pi = 1/2$ at the vertices C_i $(a_i = c_i)$; and (4) $\alpha\pi = 2\pi$ at the vertices A_i $(a_i = a_i)$. Thus, function 2 is obtained.

Obviously, the angle, a, does not vanish at points A, B but at points A_i, B_i $(i = 1, 2, 3 \ldots)$ The error was a typographical error in the manuscript.

Errata.—The following corrections should be made to the original paper:

Page 1573, Eq. 2: Should read "$\zeta^2 - 1/m^2$)" instead of "$\zeta^2 - 1/m^2$"
Page 1575, line 3: Should read "A_iB_i $(A_1B_1, A_2B_2$, etc)" instead of AB $(A_1B_1, A_2B_2$, etc)"
Page 1575, lines 10, 11, 12: Should read "A_i" instead of "A" and "B_i" instead of "B"
Page 1577, line 14: Should read "problem" instead of "problem (6)"
Page 1578, line 4: Should read "$\zeta = -a_0$" instead of "$\zeta = a_0$"
Page 1579, line 8: Should read "Eq. 12" instead of "Eq. 42"
Page 1581, Eq. 30: Should read "$\frac{\pi}{T}\cosh\frac{\pi z}{T}\frac{dz}{d\zeta}$" instead of "$\frac{k}{T}\cosh\frac{\pi z}{T}\frac{dz}{d\zeta}$"
Page 1582, Eq. 33: Should read "sn $[(2\mu + 1)k, \lambda]$" instead of "sn $[(2\mu + 1)]K, \lambda$"
Page 1583, line 23: Should read "$L = 0.9805T$" instead of "$L = 0.980ST$"

Behavior of Crossed Beams on Elastic Foundations[a]

Closure by Ami Glassman,[3] M. ASCE

The writer would like to thank Ramanathan for the interest he has shown, and the comments he has made.

The comment concerning the number of equations and the use of Digital Computers does not sound as the writer mentioned in the paper, as if the reduction of the number of the simultaneous equations is for practical use, or even with the aid of desk computers, where the number of unknowns is limited. (The example itself has been solved by a 1130 IBM Computer, and it is mentioned in the paper).

The remark that "In practice, interconnected crossed beams for foundations are used only in the form of girds . . ." does not refer to this paper, as this paper deals only with crossed beams. The discusser may find answers to this problem in Refs. 2, 4, 6, and 8. The writer approves of Eq. 16, as it was developed from Eqs. 9 and 10.

As for the experiments, the dimensions of the rubber pad were 100 cm × 100 cm. The thickness was checked as 5 cm, 10 cm, and 20 cm. The differences between the results were about 10%.

Errata.—The following corrections should be made to the original paper:

Page 7: The heading COMPUTER RESULTS should be omitted.

Seepage Through Dams with Horizontal Toe Drain[b]

Closure by Mohammad S. Moayeri,[3] A. M. ASCE

The writer wishes to thank Mayer for his discussion of the paper. Mayer correctly pointed out that in practice the horizontal permeability of the material, which is supposed to be isotropic, is usually greater than the vertical permeability. For this reason, designers of earth dams should be careful not to follow too closely the results of a mathematical study based on the assumption of homogeneous and isotropic material.

[a] January, 1972, by Ami Glassman (Proc. Paper 8626).
[3] Dir., Yotam, Advanced Design Consultants, Ltd., Haifa, Israel.
[b] May, 1972, by Mohammad S. Moayeri (Proc. Paper 8878).
[3] Assoc. Prof. of Engrg., Pahlavi Univ., Shiraz, Iran.