TECHNICAL NOTES
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PROCESS CONTROL COMPUTERS FOR EDUCATIONAL EXERCISES
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\(^a\) Discussion period closed for this paper. Any other discussion received during this discussion period will be published in subsequent Journals.
APPENDIX II.—NOTATION

The following symbols are used in this paper:

\[ e = \text{void ratio; } \]
\[ G = \text{dynamic shear modulus, in pounds per square foot (N/m}^2)\];
\[ G_s = \text{specific gravity of solid particles; } \]
\[ g = \text{acceleration of gravity = 32.2 ft per sec per sec (9.81 m/s}^2\);\]
\[ K = \frac{\sigma_h}{\sigma_v} = \text{lateral effective stress ratio; } \]
\[ N = \text{blow-count in standard penetration test; } \]
\[ P\text{-wave} = \text{compression wave; } \]
\[ q_c = \text{static, Dutch cone bearing capacity, in tons per square foot (N/m}^2\); \]
\[ S = \text{degree of saturation, as a percentage; } \]
\[ SPT = \text{abbreviation for standard penetration test; } \]
\[ S\text{-wave} = \text{shear wave; } \]
\[ t_p = \text{time of compression wave, in milliseconds; } \]
\[ t_s = \text{time of shear wave, in milliseconds; } \]
\[ v_p = \text{velocity of compression wave, in feet per second (m/s); } \]
\[ v_s = \text{velocity of shear wave, in feet per second (m/s); } \]
\[ w = \text{water content, as a percentage; } \]
\[ w_l = \text{water content at liquid limit, as a percentage; } \]
\[ w_p = \text{water content at plastic limit, as a percentage; } \]
\[ \gamma_T = \text{total unit weight, in pounds per cubic foot (N/m}^3\); \]
\[ \nu = \text{poisson’s ratio; } \]
\[ \sigma_h = \text{total horizontal effective stress, in pounds per square foot (N/m}^2\); \]
\[ \sigma_o = \text{average effective confining stress, in pounds per square foot (N/m}^2\); \]
\[ \sigma_v = \text{total effective vertical stress, in pounds per square foot (N/m}^2\). \]

INTRODUCTION

The problem of determining the position of the line of seepage in the body of an earth dam is almost as old as the basic laws of motion of ground water (Darcy’s law, 1856) itself. As early as 1883 Dupuit studied seepage flow with a free surface on a horizontal impervious boundary and, upon assuming vertical equipotential lines and a hydraulic gradient independent of depth, he obtained a parabola for the free surface. Later, Dupuit’s assumptions were used by other investigators (see Ref. 4, pp. 50-56) to study the problem of seepage through earth dams with various drain conditions.

In 1931, Kozeny (6) studied the problem of seepage through an earth dam on an impervious base with a parabolic upstream face and a horizontal underdrain. Using conformal mapping, he obtained a parabola for the free surface. Based on Kozeny’s solution, an approximate method for determining the position of the free surface in dams of trapezoidal cross sections was suggested by Casagrande (3) in 1937. He suggested a parabola for the free surface that has its focus at point B and passes through points C and E (Fig. 1), where DE is approximately 0.3H cot α and BC is given by Kozeny’s solution. Then, the entrance condition can be adjusted by sketching in the smooth arc DM normal to AD at D and tangent to the parabola at M. The position of the inflection point, M, is not clearly defined in Casagrande’s method, but, probably because of its simplicity, it has been widely used in engineering practice.

Polubarinova-Kochina (8) presents a solution for the position of the free surface which was obtained by Numerov in 1942. Although Numerov’s approach may give the solution in principle, approximations were used to obtain the resulting equations. Using Numerov’s solution, enough computations were carried out by Shankin (see Ref. 4, p. 222) to plot graphs from which the coordinates along the free surface and seepage quantities may be obtained.

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Jeppson (3) formulated the problem of seepage through earth dams in the complex potential plane and solved the finite-difference form of the Laplace equation \( \nabla^2 \phi = 0 \) in the \( \phi-\gamma \) plane by an iterative process. The free surface obtained by Jeppson was identical (at least for one example) to that obtained by Numerov everywhere except in the vicinity of the upstream face, where there was a very small deviation.

Finite-element methods of analysis have also been used (7,9) to study the problem of seepage through earth dams. In this method, a free surface is first assumed, and then the region of flow is divided into a network of triangles, the finite elements. The potential in each element is specified in terms of the values of the potential at the nodes; potential at nodes along the assumed free surface are considered unknown. Calculation of the values of potential at nodes along the free surface leads to a second approximation to the free surface.

A number of investigators (2,10; among others) have made use of the inverse hodograph and conformal mapping to obtain analytical solutions for seepage from ditches of various geometry. The principal purpose of the present study is to make use of this theory to obtain an exact solution for the problem of seepage through homogeneous and isotropic earth dams with a horizontal toe drain.

**FORMULATION OF PROBLEM**

Considered herein is a steady, incompressible, two-dimensional seepage through an earth dam on an impervious base with a horizontal toe drain (Fig. 1). In order to simplify the problem several common assumptions are made. First, the soil is taken to be homogeneous and isotropic. Second, capillary and surface tension effects are neglected. Third, the flow is assumed laminar, with negligible inertia and, thus, follows Darcy's law which may be expressed as:

\[
\vec{V} = -k \text{ grad } \left( \frac{\rho L}{\gamma} + y \right)
\]  

(1)

in which \( \vec{V} \) = velocity vector; \( \rho \) = pressure; \( \gamma \) = specific weight of the seeping fluid; \( y \) = vertical coordinate measured positively upward; and \( k \) = coefficient of permeability of the soil.

Because \( k \) is constant in a homogeneous and isotropic soil, the seepage velocity has a potential \( \phi \) such that

\[
\phi = -k \left( \frac{\rho L}{\gamma} + y \right)
\]  

(2)

and the velocity components in the \( x \) and \( y \) directions are given, respectively, by

\[
u = \frac{\partial \phi}{\partial x}; \quad u = \frac{\partial \phi}{\partial y}
\]  

(3)

Substitution of these velocity components into the equation of continuity for steady incompressible flow leads to Laplace's equation

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]  

(4)

Also, there exists a stream function, \( \psi \), which satisfies Laplace's equation and is such that

\[
\psi = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u = -\frac{\partial \psi}{\partial x}
\]  

(5)

Eqs. 3 and 5 indicate that Cauchy-Riemann conditions are satisfied in the present problem and, thus, the complex potential \( W(z) \) is given by

\[
W(z) = \phi(x,y) + i\psi(x,y)
\]  

(6)

in which \( z = x + iy \). The complex potential plane is shown in Fig. 2.

Now, the following analytic function is considered:

\[
\tau(W) = \frac{d\tau}{dW} = y\phi_\psi + i\phi_\psi
\]  

(7)

Because \( dW/dz = u - iv \), then
If both \( W \) and \( \tau \) planes are mapped conformally into the same \( \xi = \xi + i\eta \) plane by some transformation, e.g., \( \xi = f_1(W) \) and \( \tau = f_2(\xi) \), then the result will be

\[
\tau = f_2[f_1(W)] = \frac{dz}{dW} \quad \cdots \quad (10)
\]

which, upon integration, provides the geometry of the free surface.

**SOLUTION OF PROBLEM**

Using the Schwarz–Christoffel theorem, the \( w \)-plane is mapped into the upper half of the \( \xi \)-plane (Fig. 4). The mapping function is

\[
W = C_1 \int_0^\xi f(t) \, dt + D_1 \quad \cdots \quad (11)
\]

in which \( f(t) = [t(t - 1)(t - a)]^{-1/2} \quad \cdots \quad (12) \)

At point B both \( W \) and \( \xi = 0 \); therefore, \( D_1 = 0 \). At point C, where \( \xi = 1 \), the value of \( W = iq \), so that

\[
W = \frac{i\sqrt{a}}{2K(1/a)} \int_0^\xi f(t) \, dt \quad \cdots \quad (13)
\]

in which \( K \) is a complete elliptic integral of the first kind and \( q \) is the seepage quantity per unit breadth.

Putting \( \xi = \xi \), it is possible to proceed along the real axis of the \( \xi \)-plane. For \( 0 \leq \xi \leq 1 \) Eq. 13 reduces to

\[
W = iq \int \frac{\sin^{-1} \sqrt{a} \xi}{\sqrt{a}} \quad 0 \leq \xi \leq 1 \quad \cdots \quad (14)
\]

in which \( F \) is an incomplete elliptic integral of the first kind. For \( 1 \leq \xi \leq a \) the interval of integration is split at \( t = 1 \) and then the change of variable

\[
\sin \theta = \frac{a(1 - t)}{f(1 - a)} \quad \cdots \quad (15)
\]

is made (see Ref. 1, p. 597) to obtain

\[
W = iq - q F \left( \lambda, \frac{a - 1}{a} \right) \quad 1 \leq \xi \leq a \quad \cdots \quad (16)
\]

in which \( \lambda = \cos^{-1} \frac{a - \xi}{\sqrt{a(1 - a)}} \quad \cdots \quad (16) \)

At point D, where \( \xi = a \), the value of \( W \) is \( iq - kH \); therefore

\[
\text{assumed to be an equipotential line, } y_\psi = 0; \ (3) \text{ on the face AD of the dam } y_\psi = -y_\phi \text{ tan } \alpha; \ (4) \text{ on the free surface, in which } p = 0 \text{ and } \phi + ky_\phi = 0, \ y_\phi = -1/k. \text{ Therefore, the region of flow in the physical plane can be mapped into a } \tau \text{-plane (inverse hodograph) as shown in Fig. 3, in which M represents the inflection point on the free surface.}
\[
\frac{q}{kH} = \frac{K(1)}{\sqrt{a}} \frac{1}{\sqrt{\frac{a-1}{a}}} \tag{17}
\]

in which the value of \(a\) remains to be evaluated.

The Schwarz–Christoffel theorem is also used to map the \(\tau\)-plane into the same upper half of the \(\xi\)-plane. The transformation function is

\[
\tau = C_2 \int_0^\xi \frac{(\xi - m)(\xi - a)\alpha/\pi}{\sqrt{(\xi - 1)(\xi - a)}} \, d\xi + D_2 \tag{18}
\]

in which \(D_2 = 0\) because at point B both \(\tau\) and \(\xi = 0\). From Eq. 7

\[
\frac{dx}{d\xi} = \tau \frac{dW}{d\xi} \tag{19}
\]

in which \(dW/d\xi\) is obtained from Eq. 11. Therefore, integration of Eq. 19 results in

\[
x = \frac{iH - x_B}{a} \int_0^\xi G(s) \, ds + x_B \tag{20}
\]

in which \(G(s) = f(s) \int_0^s g(t) \, dt \tag{21}\)

\[
g(t) = \frac{(t - m)(t - a)\alpha/\pi}{\sqrt{(t - 1)(t - a)}} \tag{22}
\]

and the function \(f(s)\) is given in Eq. 12.

For point C Eq. 20 can be written as

\[
x_C - iy_C = \frac{(iH - x_B)}{a} \int_0^1 G(s) \, ds - x_B \tag{23}
\]

For \(0 \leq s \leq 1\), the integrand function \(g(t)\) is first written in the form

\[
g(t) = \frac{-1}{i(-1)^n} \left[ \frac{m - t}{(a - t)^n} \sqrt{(1 - t)} \right] \tag{24}
\]

in which \(n = (\alpha/\pi) + (1/2)\). Then, the change of variable \(t = \sin^2 \theta\) is made to obtain

\[
\int_0^1 G(s) \, ds = \frac{4}{i(-1)^n} \int_0^{\pi/2} \Lambda(\theta)(a - \sin^2 \theta)^{-1/2} \, d\theta = \frac{4}{i(-1)^n} (I_4) \tag{25}
\]

in which the function, \(\Lambda\), is defined as

\[
\Lambda(\beta) = \int_0^\beta (m - \sin^2 \theta)(a - \sin^2 \theta)^{-n} \, d\theta \tag{26}
\]

For \(1 \leq s \leq a\), the interval of integration is split at \(t = 1\) and the change of variable \(t = \cosh^2 \theta\) is made to obtain

\[
\int_1^a G(s) \, ds = \frac{2}{i(-1)^n} \Lambda(\frac{\pi}{2}) \int_1^a [s(s - 1)(a - s)]^{-1/2} \, ds - \frac{2}{i(-1)^n} \int_1^a [s(s - 1)(a - s)]^{-1/2} \Omega(\cosh^{-1} \sqrt{s}) \, ds \tag{27}
\]

in which the function, \(\Omega\), is defined as

\[
\Omega(\beta) = \int_0^\beta (m - \cosh^2 \theta)(a - \cosh^2 \theta)^{-n} \, d\theta \tag{28}
\]

But, from Eqs. 13 and 15, when \(\xi = a\)

\[
\int_1^a [s(s - 1)(a - s)]^{-1/2} \, ds = \frac{2}{\sqrt{a}} K\left(\sqrt{\frac{a - 1}{a}}\right) \tag{29}
\]

Substituting Eq. 29 into Eq. 27 and making the change of variable \(s = \cosh^2 \theta\) in the last integral at the right of Eq. 27, the following is obtained:

\[
\int_1^a G(s) \, ds = \frac{4}{i(-1)^n} K\left(\sqrt{\frac{a - 1}{a}}\right) \Lambda(\frac{\pi}{2}) - \frac{4}{i(-1)^n} I_4(\cosh^{-1} \sqrt{a}) \tag{30}
\]

in which \(I_4(\beta) = \int_0^\beta (a - \cosh^2 \theta)^{-1/2} \Omega(\theta) \, d\theta \tag{31}\)

Substituting Eqs. 25 and 30 into Eq. 23, separating real and imaginary parts, and considering the fact that \(y_C = 0\), leads to Eq. 32

\[
\frac{x_B}{H} = \sqrt{a} \left[ I_4(\cosh^{-1} \sqrt{a}) - I_1 \right] \tag{32}
\]

\[
\frac{\Lambda(\frac{\pi}{2}) K\left(\sqrt{\frac{a - 1}{a}}\right)}{\Lambda(\frac{\pi}{2}) K\left(\sqrt{\frac{a - 1}{a}}\right)}
\]

and

\[
\frac{x_C}{H} = \frac{x_B}{H} + \sqrt{a} I_4 \tag{33}
\]

Along the free surface, where \(\xi \) changes from 1 to \(a\), Eq. 20 can be written as

\[
\frac{x}{H} + i \frac{y}{H} = \left( i - \frac{x_B}{H} \right) \int_0^\xi G(s) \, ds + \frac{x_B}{H} \tag{34}
\]

When Eqs. 25, 30, and 33 are substituted into Eq. 34, and real and imaginary parts separated, the following equations will be obtained for the coordinates of the free surface

\[
\frac{x}{H} = \frac{x_C}{H} - \sqrt{a} I_4(\cosh^{-1} \sqrt{a}) \tag{35}
\]

\[
1 \leq \xi \leq a \tag{35}
\]
\[
\frac{y}{H} = \frac{F\left(\lambda, \sqrt{\frac{a-1}{a}}\right)}{K\left(\sqrt{\frac{a-1}{a}}\right)}; \quad 1 \leq \xi = a \quad \text{............................ (36)}
\]

in which \(\lambda\) is given in Eq. 16.

To calculate the unknown parameters \(m\) and \(a\), values of \(x\) and \(\xi\) at point \(A\) are used in Eq. 20, that is

\[
\cot \alpha + \frac{x_B}{H} = \left(\frac{x_B}{H} - \frac{x}{H}\right) \left(1 + \int_{0}^{a} G(s) \, ds \right) \quad \text{............................ (37)}
\]

For \(1 \leq s \leq \infty\) the interval of integration is split at \(t = 1\) and at \(t = a\), so that

\[
\int_{a}^{\infty} G(s) \, ds = -\left[\frac{2}{\sqrt{(1-a)^2}} \Lambda\left(\frac{\pi}{2}\right)\right. \right.
\]

\[
+ \left. \frac{2}{1-a^2} \Omega \left(\cosh^{-1} \sqrt{a}\right) \right]\int_{a}^{\infty} f(s) \, ds + I_3 \quad \text{............................ (38)}
\]

in which

\[
I_3 = \int_{a}^{\infty} f(s) \int_{a}^{s} g(t) \, dt \, ds \quad \text{............................ (39)}
\]

and

\[
\int_{a}^{\infty} f(s) \, ds = \frac{2}{\sqrt{a}} \Lambda\left(\frac{1}{\sqrt{a}}\right) \quad \text{............................ (40)}
\]

and the functions \(f(s)\) and \(g(t)\) are given in Eqs. 12 and 22, respectively. Substituting Eqs. 25, 30, and 38 into Eq. 37, separating real and imaginary parts, and making use of the relationship \((-1)^n = \cos(n\pi) + i \sin(n\pi)\), the following is obtained:

\[
\Lambda\left(\frac{\pi}{2}\right) \tan \alpha + \Omega \left(\cosh^{-1} \sqrt{a}\right) = 0 \quad \text{............................ (41)}
\]

As is seen from Eqs. 26 and 28, the values of the integral functions \(\Lambda\) and \(\Omega\) depend upon both parameters \(m\) and \(a\). Fortunately, for a given value of \(\alpha\) between 0 and \(\pi/2\) and for a given value of \(a > 1\), there is only one value of \(m\) which satisfies Eq. 41 and is such that \(1 < m < a\). Therefore, for each value of \(\alpha\), solution of Eq. 41 leads to a single curve describing the relationship between \(a\) and \(m\) for that \(\alpha\). Then, values of \(m\) and \(a\) taken from this curve can be used in Eq. 32 to calculate the corresponding value of \(x_B/H\).

The case \(\alpha = \pi/2\): Because of the absence of an inflection point on the free surface (Fig. 5) formulation and solution of the problem for \(\alpha = \pi/2\) is somewhat different from that for \(0 < \alpha < \pi/2\). For this case the inverse velocity plane and the auxiliary plane of \(\xi\) are given in Figs. 6 and 7, respectively, while the complex potential plane remains unchanged. The transformation function by which the \(W\)-plane is mapped into the upper half of the \(\xi\)-plane is given by

\[
W = \frac{iq}{\int_{0}^{1} r(t) \, dt} \int_{0}^{\xi} r(t) \, dt \quad \text{............................ (42)}
\]
in which $r(t) = [t(t - 1)(t + a)]^{-1/a}$  \(43\)

The use of the coordinates of point A (Fig. 5) in Eq. 42 results in the following equation for the quantity of seepage:

$$\frac{q}{kH} = \frac{K}{\sqrt{1 + a}}$$  \(44\)

The mapping function of the $\tau$-plane is

$$\tau = C_s \int_0^\zeta [\zeta(\zeta(1)]^{-1/a} d\zeta = C_s \ln (1 - 2\zeta - 2\sqrt{\zeta^2 - \zeta})$$  \(45\)

so that $z = C_s \int_0^\zeta h(s) \, ds + x_B$  \(46\)

in which $h(s) = r(s) \ln (1 - 2s - 2\sqrt{s^2 - s})$,  \(47\)

and $C_s = \int_0^1 h(s) \, ds$  \(48\)

and $r(s)$ is given in Eq. 43. For point C, at which $z = x_C$ and $\zeta = 1$, Eq. 46 can be written as

$$\frac{x_C}{H} = \frac{x_B}{H} \left[ 1 + \frac{\int_0^1 h(s) \, ds}{\int_{-1}^0 h(s) \, ds} \right]$$  \(49\)

For $0 \leq s \leq 1$, the integrand function is first written in the form

$$h(s) = \frac{\ln (1 - 2s - 2\sqrt{s^2 - s})}{i \sqrt{(1 - s)(s + a)}}$$

$$= \tan^{-1} \left( \frac{2\sqrt{s - s^2}}{1 - 2s} \right) \left[ s(1 - s)(s + a) \right]^{-1/a}$$

and then the change of variable $s = \sin^2 \theta$ is made to obtain

$$\int_0^1 h(s) \, ds = -4 \int_0^{\pi/2} \theta(a + \sin^2 \theta)^{-1/a} \, d\theta = -4I_s$$  \(50\)

For $-1 \leq s \leq 0$, the change of variable $s = -a \cos^2 \theta$ may be used to obtain

$$\int_{-a}^0 h(s) \, ds$$

$$= -4 \int_0^{\pi/2} (1 + a \cos^2 \theta)^{-1/a} \sinh^{-1}(\sqrt{a} \cos \theta) \, d\theta = -4I_s$$  \(51\)

When Eqs. 50 and 51 are substituted into Eq. 49, the following equation is obtained:

$$\frac{x_C}{H} = \frac{x_B}{H} \left( 1 + \frac{I_s}{I_s} \right)$$  \(52\)

At point D, where $\zeta = \infty$, the value of $z$ is $iH$; therefore

$$t = \frac{x_B}{H} \left[ \frac{1}{1} + \int_{-a}^0 h(s) \, ds \right]$$  \(53\)

For $1 \leq \zeta \leq \infty$, the integrand function can be first written as

$$h(s) = [\ln (2s + 2\sqrt{2s^2 - s} - 1) - i\pi] [s(s - 1)(s + a)]^{-1/a}$$

and then the change of variable $s = \sec^2 \theta$ may be made to obtain

$$\int_1^\zeta h(s) \, ds = -4i \left( \frac{1}{\sqrt{1 + a}} \right) F \left( \cos^{-1} \frac{1}{\sqrt{1 + a}} \right)$$

$$+ 4I_s \left( \cos^{-1} \frac{1}{\sqrt{1 + a}} \right) 1 \leq \zeta \leq \infty$$  \(54\)

---

**FIG. 8.—VARIATION OF $a$ AND $m$ WITH $\alpha$ AND $x_B/H$**
in which \( I_\alpha(\beta) = \int_\beta^\infty \ln(\sec \theta + \tan \theta)(1 + a \cos^2 \theta)^{-1/2} d\theta \) \ldots (55)

when \( \xi = \infty \), Eq. 54 reduces to
\[
\int_1^\infty h(s) \, ds = -\frac{2ni}{\sqrt{1 + a}} K\left(\frac{a}{\sqrt{1 + a}}\right) + 4I_\alpha\left(\frac{\pi}{2}\right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (56)
\]

Substituting Eqs. 50, 51, and 56 into Eq. 53 and separating real and imaginary parts, leads to the following equation for \( x_B/H \):
\[
\frac{x_B}{H} = \frac{2}{\pi} \sqrt{1 + a} I_\delta K\left(\frac{a}{\sqrt{1 + a}}\right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (57)
\]

so that, Eq. 52 may now be written as
\[
\frac{x_C}{H} = \frac{x_B}{H} + \frac{2}{\pi} \left[ \frac{1}{\sqrt{1 + a}} \right] \frac{F}{K}\left(\frac{a}{\sqrt{1 + a}}\right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (58)
\]

FIG. 9.—VARIATION OF \( q/kH \) WITH \( \alpha \) AND \( x_B/H \)

FIG. 10.—VARIATION OF \( L/H \) WITH \( \alpha \) AND \( x_B/H \)

For the free surface, where \( \xi \) changes from 1 to \( \infty \), Eq. 46 can be written as
\[
\frac{x}{H} + i\frac{y}{H} = \frac{x_C}{H} - \frac{x_B}{H} \int_0^\xi h(s) \, ds \frac{1}{4I_\delta} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (59)
\]

When Eqs. 54 and 57 are substituted into Eq. 59 and real and imaginary parts separated, the following equations are found for the coordinates of the free surface:
\[
\frac{x}{H} = \frac{x_C}{H} - \frac{2}{\pi} \sqrt{1 + a} I_\delta \left(\cos^{-1}\frac{1}{\xi}\right) \quad 1 \leq \xi \leq \infty \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (60)
\]

and
\[
\frac{y}{H} = F \left(\cos^{-1}\frac{1}{\xi}\right) - \frac{2}{\pi} \sqrt{1 + a} I_\delta \left(\frac{A}{\sqrt{1 + a}}\right) \quad 1 \leq \xi \leq \infty \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (61)
\]

in which \( I_\delta \) is given in Eq. 55.
NUMERICAL CALCULATIONS AND RESULTS

Formulas were programmed in Fortran for an IBM 1130, using Simpson's rule to evaluate the integrals. Because Eqs. 32 and 41 are highly nonlinear in \( a \) and \( m \), direct solution of these equations was not possible and a special method of successive approximations, similar to Newton's method, was used to solve Eq. 41. Starting with an assumed value of \( m_N \), Eq. 62

\[
R_N = \left( \frac{\sqrt{a}}{2} \right) \tan \alpha + \Omega (\cosh^{-1} \sqrt{a}) \tag{62}
\]

was used to calculate \( R_N \) for each given values of \( \alpha \) and \( a \). A correction term, \( \Delta m_N \), was obtained from

\[
\Delta m_N \equiv - \frac{R_N}{\frac{\partial R}{\partial m}} \tag{63}
\]

in which, from Eqs. 62, 26, and 28

\[
\frac{\partial R}{\partial m} = \tan \alpha \int_0^{\pi/2} (a - \sin^2 \theta)^{-n} \, d\theta
+ \int_0^{\cosh^{-1} \sqrt{a}} (a - \cosh^2 \theta)^{-n} \, d\theta \tag{64}
\]

Then \( \Delta m_N \) was added to \( m_N \) to obtain a better approximation \( m_{N+1} \). Using this method, the exact value of \( m \), for \( R = 0 \), was obtained at the third approximation. Then the value of \( m \), as obtained herein, was used in Eq. 32 to calculate the corresponding value of \( x_B/H \). The results of calculations, using \( \alpha = 15^\circ \), \( 30^\circ \), \( 45^\circ \), and \( 60^\circ \), have been plotted in Fig. 8, from which values of \( a \) and \( m \) can be obtained for a given physical condition.

Knowing the values of \( a \) from Fig. 8, Eq. 17 was used to calculate the quantities of seepage. A plot of \( q/kH \) against \( x_B/H \) for different values of \( \alpha \) is given in Fig. 9. Likewise, knowing the values of \( a \) and \( m \) from Fig. 8, Eq. 33 was used to plot the length of the drain, \( L/H \), against \( x_B/H \) for different values of \( \alpha \) (Fig. 10). The variations of \( q/kH \) and \( L/H \) with \( x_B/H \) for \( \alpha = \pi/2 \) have been plotted in Fig. 12, while the relationship between \( a \) and \( x_B/H \) for this case is given in Fig. 11.

In order to compare the results obtained by the present study and those obtained by other investigators, \( q/kH \) and \( L/H \) were also calculated for different values of \( x_B/H \) and \( \alpha \), using Casagrande's (3) method and the Numerov equations (8, pp. 243-246). The relationship between \( q/kH \) and \( x_B/H \) are compared in Fig. 9, which also contains the results obtained by Jeppson (5) for \( \alpha = \pi/4 \). As is seen from Fig. 9, Casagrande's method can be considered as a good approximation only when \( x_B/H > 1 \) and \( \alpha > \pi/4 \). Also, Fig. 9 indicates that Numerov's solution gives values of \( q/kH \) larger than those obtained by the present study and the difference is smaller for larger values of \( \alpha \). The variations of \( L/H \) with \( x_B/H \) for different values of \( \alpha \) are compared in Fig. 10; here again Numerov's solution gives larger values of \( L/H \) at each value of \( x_B/H \), but the difference decreases as the angle, \( \alpha \), increases.

A comparison between the free surface profile obtained by the present study and those obtained by Casagrande's method and Numerov's equations is made in Fig. 1. Fig. 1 indicates that for \( \alpha = \pi/4 \) the free surface profile obtained
by Casagrande's method follows the result of the present study better than the profile obtained by the Numerov equations. This, of course, is not generally true for all values of $\alpha$, as Figs. 9 and 10 indicate that Casagrande's method is not a good approximation when values of $\alpha$ and $x_B/H$ are smaller than, respectively, 45° and one. Fig. 1 also indicates that the coordinates along the free surface calculated by the Numerov equations are larger than those obtained by the present study, but the difference is very small in the vicinity of the drain.

The position of the point of inflection on the free surface depends upon both variables $x_B/H$ and $\alpha$. For a given value of $\alpha$, the inflection point, M, approaches the point, D (Fig. 1) as $x_B/H$ decreases, or as $m/a$, in Fig. 8, approaches unity. The inflection point disappears when $m = a$ and this occurs at a definite value of $x_B/H$ for a given value of the angle $\alpha$, as shown in Fig. 13. Specifically, points above the curve in Fig. 13 correspond to the conditions where there is always an inflection point on the free surface, while a point below this curve represents a condition where an inflection point is absent on the free surface. The values of $x_B/H$ at which the inflection point on the free surface vanishes were calculated by Polubarinova-Kochina (8, p. 42) for values of $\alpha$ equal to $\pi/4$ and $\pi/16$ with the following results: for $\alpha = \pi/4$, $x_B/H = 1/(1.16 \sin \alpha)$ - $\cot \alpha = 0.22$; and for $\alpha = \pi/16$, $x_B/H = 1/(1.15 \sin \alpha)$ - $\cot \alpha = -0.72$. As is seen in Fig. 13, these results, especially for $\alpha = \pi/16$, do not agree with the results of the present study.

Fig. 13.—VALUES OF $\alpha$ AND $x_B/H$ FOR WHICH $m/a - 1$

It is interesting that for $\alpha = \pi/2$, as is seen in Figs. 5 and 12, the results of the present study differ very little from those obtained by Numerov.

### SUMMARY AND CONCLUSIONS

Inverse hodograph and conformal mapping is used to obtain an exact solution to the problem of seepage through homogeneous and isotropic earth dams with a horizontal toe drain. Earth dams with the angle between the upstream face and horizontal equal to 15°, 30°, 45°, 60°, and 90° are considered. The results are partly given in the form of graphs, from which the quantity of seepage, the length of the drain, and the unknown parameters, that are necessary for calculation of the coordinates along the free surface, can be obtained.

A comparison is made between the results of the present study and those obtained by other investigators, which indicates that Casagrande's (3) method of finding the approximate position of the free surface gives a good result only when the values of $x_B/H$ and $\alpha$ are larger than one and 45°, respectively. Also, the quantity of seepage and the coordinates along the free surface calculated by the Numerov equations (8, pp. 243-246) are more accurate for larger values of the angle $\alpha$.

The method can be used to study the problem of seepage through earth dams of different boundary configurations and drain conditions.

### APPENDIX I.—REFERENCES

10. Vedenkov, V. V., "Seepage from Channels," (Versickerungen aus Kanalen), Wasserwirtschaft und Wasserwirtschaft, No. 11, 12, 13, 1934.
APPENDIX II.—NOTATION

The following symbols are used in this paper:

- \( A \) = subscript defining intersection point of upstream face of dam with base;
- \( a \) = nonphysical parameter resulting from Schwarz-Christoffel transformation;
- \( B \) = subscript defining upstream end of toe drain;
- \( C \) = subscript defining downstream end of free surface;

\( C_1, C_2, C_3, C_4, C_5 \) = constants of integration;

\( D \) = subscript defining upstream end of free surface;

\( D_1, D_2, D_3 \) = constants of integrations;

\( E \) = subscript defining point on surface of reservoir at which \( x = - 0.3 H \cot \alpha \);

\( F \) = incomplete elliptic integral of first kind;

\( f_1, f_2 \) = transformation functions;

\( f(t) \) = integrand function (see Eq. 13);

\( G(s) \) = integrand function (see Eq. 21);

\( g(t) \) = integrand function (see Eq. 22);

\( H \) = total head on dam;

\( h(s) \) = integrand function (see Eq. 47);

\( J_1, J_2, J_3, J_4, J_5, J_6 \) = integral functions (see Eqs. 25, 31, 39, 50, 51, and 55, respectively);

\( i \) = imaginary unit;

\( K \) = complete elliptic integral of first kind;

\( k \) = coefficient of permeability;

\( L \) = length of toe drain;

\( M \) = subscript defining inflection point on free surface;

\( m \) = nonphysical parameter resulting from Schwarz-Christoffel transformation;

\( N \) = order of approximation;

\( n \) = \( \alpha/\pi + 1/2 \);

\( o \) = subscript defining origin of x-y coordinates;

\( p \) = pressure;

\( q \) = seepage quantity per unit breadth;

\( R_N \) = residual at \( N \)th approximation (see Eq. 62);

\( r(t) \) = integrand function (see Eq. 43);

\( s \) = dummy variable of integration;

\( t \) = dummy variable of integration;

\( u \) = horizontal component of velocity;

\( v \) = velocity vector;

\( V \) = velocity vector;

\( W \) = complex potential;

\( x \) = horizontal distance from \( y \) axis;

\( x_B \) = distance from point B to origin;

\( x_C \) = distance from point C to origin;

\( y \) = vertical distance from \( x \) axis;

\( y_\delta \) = real part of \( \tau \);

\( y_\delta \) = imaginary part of \( \tau \);

\( z = x + iy \);

\( \alpha \) = angle between upstream face of dam and horizontal base;

\( \beta \) = dummy variable;

\( \gamma \) = specific weight of liquid;

\( \Delta m \) = correction term for \( m \);

\( \xi \) = auxiliary function;

\( \eta \) = imaginary part of \( \xi \);

\( \theta \) = dummy variable of integration;

\( \Lambda \) = integral function (see Eq. 26);

\( \lambda = \cos^{-1} \frac{\sqrt{(a - \xi)/\xi (a - 1)} \sqrt{(a - \xi)/\xi (a - 1)}} \);  

\( \xi \) = real part of \( \xi \);

\( \tau = y_\delta + iy_\delta \);

\( \phi \) = velocity potential, real part of \( W \);

\( \psi \) = stream function, imaginary part of \( W \); and

\( \Omega \) = integral function (see Eq. 28).
APPENDIX.—REFERENCES


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Seepage Through Dams with Horizontal Toe Drain

Discussion by Armand Mayer, 2 Hon. M. ASCE

The object of this discussion is to add the writer’s experience to the mathematical treatment of seepage through earth dams. The writer had the opportunity of planning and supervising construction of three earth dams with horizontal toe drains. The material in the three cases was a homogenous, slightly silty sand, highly compacted, with very good laboratory supervision during construction. When finished it appeared that the material, which was supposed to be homogenous, had a horizontal permeability more than 10 times greater than the vertical one, so that despite the toe drain, water filtered horizontally and appeared in several locations on the downstream side of the dam.

There was no difficulty in draining the outlets but the writer became very skeptical about the hypothesis of homogeneous dams and the effectiveness of plain toe drains. Since then the writer has always given the toe drains an L-shape to eliminate all horizontal seepage thru the dam. Perhaps this experience will help designers who might be tempted to follow too closely the results of a mathematical study, based on a hypothesis which is very seldom, if ever, verified in nature.

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Discussion by Narayan Nayak, 1 A. M. ASCE

The writer has written a Foundation Design Manual for the University of Panama, Panama, which will be published in Spanish. Based on this experience,


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give a clear insight to the problem. Consequently, they are preferable in case the model of homogeneous pervious stratum is considered as satisfactory or the lack of sufficient field measurements makes it necessarily acceptable. The writer believes that, in these cases, the application of the more accurate analytical methods is justified since the addition of computational errors to the modeling errors is so avoided without excessive work in comparison to less accurate methods. It must also be noted that some problems of determining the flow characteristics for structures founded in an heterogeneous pervious stratum can be reduced to one for a homogeneous pervious stratum. Polubarinova-Kochina has suggested such a method in case the pervious stratum is composed of two layers of different permeability (4,8).

A question raised in the discussion concerns the derivation of mapping functions 1 and 2. The functions, 1 and 2, immediately result from Christoffel-Schwarz transformation (Eq. 54) on the basis of Figs. 1(a), 1(b), 1(c). For the rectangle \( \omega \) [Fig. 1(b)] \( \alpha_1 \pi = \alpha_2 \pi = \alpha_3 \pi = \alpha_4 \pi = \alpha_5 \), \( a_5 = -1/m \), \( a_2 = -1 \), \( a_1 = -1 \), \( a_4 = +1/m \). Consequently, function 1 is obtained. For the polygon, \( \gamma \), the angles, \( \alpha \pi \), take the following values: (1) \( \alpha \pi = 0 \) at the vertices \( C_\omega (a_1 = -1/m) \), \( D_\omega (a_2 = -1/m) \); (2) \( \alpha \pi = 1/2 \) at the vertices \( A (a_2 = -1), B \) (\( a_4 = +1 \)); (3) \( \alpha \pi = 1/2 \) at the vertices \( C_i (a_i = c_i) \); and (4) \( \alpha \pi = 2\pi \) at the vertices \( A_\gamma (a_i = a_i) \). Thus, function 2 is obtained.

Obviously, the angle, \( \theta \), does not vanish at points \( A, B \) but at points \( A_\gamma, B_\gamma \) (\( i = 1, 2, 3 \ldots \)). The error was a typographical error in the manuscript.

Errata.—The following corrections should be made to the original paper:

Page 1573, Eq. 2: Should read "\( \zeta^2 - 1/m^2 \)" instead of "\( \zeta^2 - 1/m^3 \)"

Page 1575, line 3: Should read "\( A_iB_i(A, B, A_2, B_2, etc.)\)" instead of "\( AB (A, B, A_2, B_2, etc.)\)"

Page 1575, lines 10, 11, 12: Should read "\( A_1 \)" instead of "\( A \)" and "\( B_1 \)" instead of "\( B \)"

Page 1577, line 14: Should read "problem" instead of "problem (6)"

Page 1578, line 4: Should read "\( \xi = a_0 \)" instead of "\( \xi = a_0 \)"

Page 1579, line 8: Should read "Eq. 12" instead of "Eq. 42"

Page 1581, Eq. 30: Should read "\( \frac{\pi}{T} \cosh \frac{\pi z}{T} \frac{dz}{d\zeta} \)" instead of "\( \frac{\pi}{T} \cosh \frac{\pi z}{T} \frac{dz}{d\zeta} \)"

Page 1582, Eq. 33: Should read "\( s \eta \left[ (2\mu + 1)k, \lambda \right] \)" instead of "\( s \eta \left[ (2\mu + 1) \right] K, \lambda \)"

Page 1583, line 23: Should read "\( L = 0.9805ST \)" instead of "\( L = 0.980ST \)"

The writer would like to thank Ramanathan for the interest he has shown, and the comments he has made.

The comment concerning the number of equations and the use of Digital Computers does not sound as the writer mentioned in the paper, as if the reduction of the number of simultaneous equations is for practical use, or even with the aid of desk computers, where the number of unknowns is limited. (The example itself has been solved by a 1130 IBM Computer, and it is mentioned in the paper).

The remark that "In practice, interconnected crossed beams for foundations are used only in the form of girds . . ." does not refer to this paper, as this paper deals only with crossed beams. The discussor may find answers to this problem in Refs. 2, 4, 6, and 8. The writer approves of Eq. 16, as it was developed from Eqs. 9 and 10.

As for the experiments, the dimensions of the rubber pad were 100 cm \( \times \) 100 cm. The thickness was checked as 5 cm, 10 cm, and 20 cm. The differences between the results were about 10%.

Errata.—The following corrections should be made to the original paper:

Page 7: The heading COMPUTER RESULTS should be omitted.

SEEPAGE THROUGH DAMS WITH HORIZONTAL TOE DRAIN

Closure by Mohammad S. Moayeri, A. M. ASCE

The writer wishes to thank Mayer for his discussion of the paper. Mayer correctly pointed out that in practice the horizontal permeability of the material, which is supposed to be isotropic, is usually greater than the vertical permeability. For this reason, designers of earth dams should be careful not to follow too closely the results of a mathematical study based on the assumption of homogeneous and isotropic material.

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*Assoc. Prof. of Engrg., Pahlavi Univ., Shiraz, Iran.
In the paper, inverse hodograph and conformal mapping is used to obtain an exact solution to the problem of seepage through homogeneous and isotropic earth dams with horizontal toe drain. One of the results is the variation of the effective drain length, \( L/H \), with \( \alpha \) and \( X_B/H \), which is given in Fig. 10. The writer believes that practicing engineers should not be afraid of using this result, however, to make the horizontal toe drain more effective, a vertical section may be added to the drain at point C, as shown in Fig. 14.

Errata.—The following corrections should be made to the original paper:

Page 465, Eq. 14: Should read \[ W = iq \frac{F\left(\sin^{-1}\sqrt{\xi}, \frac{1}{\sqrt{a}}\right)}{K\left(\frac{1}{\sqrt{a}}\right)} \]

instead of \[ W = iqF \frac{\left(\sin^{-1}\sqrt{\xi}, \frac{1}{\sqrt{a}}\right)}{K\left(\frac{1}{\sqrt{a}}\right)} \]

Page 465, Eq. 15: Should read \[ W = iq - q \frac{F\left(\lambda, \sqrt{\frac{a-1}{a}}\right)}{K\left(\frac{1}{\sqrt{a}}\right)} \]

instead of \[ W = iq - qF \frac{\left(\lambda, \sqrt{\frac{a-1}{a}}\right)}{K\left(\frac{1}{\sqrt{a}}\right)} \]

In the opinion of this discusser, the portion of the proposed design manual titled “PENETROMETER TESTS” needs revision.

The single paragraph under this title mentions only penetrometer tests in which the engineer uses “counting the blows or energy required” as the measure of soil resistance. Engineers refer to this type of test as a “dynamic penetration test.” The Peck cone penetrometer shown in Fig. 3 shows one type of dynamic penetrometer. Dynamic penetrometers usually do not provide a positive means for eliminating the drive pipe friction part of the soil resistance measurement.

The paragraph then goes on to mention the cone-type penetrometers extensively used in Holland, especially for determining relative density of granular soil. The Dutch use, almost exclusively, the “static” cone penetration test, wherein the operator does not drive the cone with a hammer but instead advances it with a constant speed of 1 cm/s to 2 cm/s. The Dutch penetrometer equipment also includes casing protection against buckling and soil friction for the rods used to advance the cone tip, or they place electric resistance sensing elements within the tip (21). Dutch engineers have in the past rarely used cone penetration tests to determine relative density. They do use extensively the static cone penetration test to detect the presence and bearing strength of sand layers, for the purpose of designing pile foundations.

The results of relatively recent research now permit engineers to use the static cone penetration test for more than pile design purposes. In cohesionless soils an engineer may use it to control compaction, estimate relative density, and estimate settlement. In cohesive soils the test permits estimating shear strength and compressibility. Because of its speed, economy, and the continuous or near-continuous vertical resistance profile data obtained, many engineers find these tests particularly useful to evaluate the uniformity of soil conditions at a site. The recent book by Sangerlatt (22) details these and other uses for static cone penetrometer test data.

The paragraph under discussion also notes that the use of penetrometer tests without borings “is not considered good practice” because of the absence of samples for inspection. This statement requires clarification or amplification.

In 1949, Hvorslev (6) notes “soil soundings should therefore be supplemented by borings unless (discussers’s emphasis) the soil types found in the area under investigation already are known.” Engineers now have the friction-cone penetrometer tip, not available in 1949, which permits the measurement of local soil