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DIVISION NAME CHANGE

The Technical Activities Committee, at its July 9-10, 1973 meeting, held in Tulsa, Oklahoma, approved the change in name of the Soil Mechanics and Foundations Division to the Geotechnical Engineering Division. However, we are continuing to use the "old" name for the Journals for the balance of 1973. The January 1974 issue will carry the new name.

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DISCUSSION
Proc. Paper 10038

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TECHNICAL NOTES
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Accuracy of Equilibrium Slope Stability Analysis

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Introduction

Although limit equilibrium procedures of slope stability analysis have been widely and successfully used, these methods are subject to criticism on theoretical grounds for three reasons:

1. Arbitrary assumptions are employed so that the normal stress on the shear surface may be determined using only the conditions of static equilibrium, without consideration of the stress-strain characteristics of the soil; these arbitrary assumptions most frequently concern the locations or directions of side forces on slices.
2. Most of the equilibrium methods, including those most widely used and considered to be most accurate [Bishop's Simplified Method (1), Morgenstern and Price's Method (6), and Junbu's Generalized Procedure of Slices (4)] involve the assumption that the factor of safety is the same for every slice, even though there is no reason to except this to be true except at failure, when the factor of safety is equal to one for every slice.
3. Some of the equilibrium methods, including the Ordinary Method of Slices or Swedish Circle Method, Bishop's Simplified Method, and the wedge methods based on force equilibrium, do not satisfy all the conditions of equilibrium.

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The study described in this paper was made to examine the effect of these factors by comparing the results of limit equilibrium analyses with the results obtained by a completely independent analysis procedure, which involves none of these assumptions or shortcomings.

Limit Equilibrium Procedures

The definition of the factor of safety most frequently used in the equilibrium procedures of slope stability analysis is

$$F = \frac{s}{\tau} \qquad (1)$$

in which s = the shear strength of the soil; and τ = the shear stress required for equilibrium. It may be shown that the definition of F given by Eq. 1 is equivalent to the definition employed in the Ordinary Method of Slices, where the factor of safety is defined as the ratio of the resisting moment to the overturning moment.

The value of the shear strength, s at any point on a potential shear surface is dependent on the normal stress, σ at that point, as shown by

$$s = c + \sigma \tan \phi \qquad (2)$$

in which c and ϕ = the Mohr-Coulomb shear strength parameters. Therefore, except for the special case of $\phi = 0$, the normal stress on the shear surface must be known before the shear strength may be determined. The problem of determining the distribution of the normal stress on the shear surface is statically indeterminate, i.e., the problem involves more unknowns than there are equations of equilibrium. In order to solve the problem, it is necessary to increase the number of equations, or reduce the number of unknowns. The number of equations may be increased by considering the stress-strain characteristics of the soil and the requirements of compatibility of deformations. Alternatively, the number of unknowns may be reduced by making assumptions. The latter is the method used in the limit equilibrium analysis procedures.

All limit equilibrium methods of slope stability analysis employ assumptions to reduce the number of unknowns to be equal to the number of equations of equilibrium. However, not all of the equilibrium methods satisfy the same conditions of equilibrium. Some methods, like Janbu's Generalized Procedure of Slices and Morgenstern and Price's procedure, satisfy all conditions of equilibrium. This provides two equations of force equilibrium and one equation of moment equilibrium for each slice, and $3N$ equations in total, where N is the number of slices. Other methods, like Bishop's Simplified Method and the Ordinary Method of Slices, do not satisfy all conditions of equilibrium. Bishop's Simplified Method satisfies vertical equilibrium for each slice and overall moment equilibrium, but does not satisfy horizontal equilibrium or moment equilibrium for each slice; as a result, the number of equations is $N + 1$. The Ordinary Method of Slices satisfies only overall moment equilibrium, but not moment or force equilibrium for individual slices; as a result, there is just one equation.

Comparative studies of the equilibrium methods by Whitman and Bailey (7) and by Wright (8) have led to the following conclusions.

1. The values of the factor of safety calculated using Janbu's Generalized Procedure of Slices and Morgenstern and Price's Method are very nearly the same. Any method which satisfied all conditions of equilibrium was found to give virtually the same value of factor of safety, for any reasonable set of assumptions employed.
2. The values of the factor of safety calculated by Bishop's Simplified Method are generally comparable to those calculated by methods satisfying all conditions of equilibrium. The difference was found to vary from 0% to 6% for a wide variety of conditions of slope angle, shear strength, and pore pressure.
3. The values of factor of safety calaculated by the Ordinary Method of Slices are generally smaller than those calculated by methods satisfying all conditions of equilibrium, and are also smaller than those calculated by Bishop's Simplified Method. The difference was found to increase with increasing values of the angle subtended by the slip circle, and with increasing magnitude of the pore pressure. For extreme conditions the value of the factor of safety calculated by the Ordinary Method of slices may be only half as large as the value calculated by a method that satisfies all conditions of equilibrium, or by Bishop's Simplified Method.

The fact that the value of factor of safety calculated by Bishop's Simplified Method and by methods satisfying all conditions of equilibrium are nearly the same is considered a strong indication that these methods give the "right answer." However, as all of these methods share many common features, the determination that they give nearly the same value of F does not necessarily indicate that all of the methods are accurate; it might only indicate that they are all about equally inaccurate.

The studies described in this paper were performed to examine the accuracy of the equilibrium methods of slope stability analysis by comparing normal stresses and factors of safety calculated by Bishop's Simplified Method with values calculated by a completely independent method. This method consisted of using the internal stresses determined by performing finite element analyses of a number of slopes to determine: (1) The variations of normal stress and factor of safety along the shear surface; and (2) the overall factor of safety for each slope. In each case comparisons were made for the critical circle as determined by Bishop's Simplified Method.

Basis of Comparison—$\lambda_{c\phi}$

The stability of a homogeneous slope, similar to the ones considered in this study, depends on the slope height H, its inclination, β, the unit weight of the soil, γ, and the values of the shear strength parameters, c and ϕ. Janbu (4) has shown that the effects of many of these factors may be combined, and the study of these effects may be simplified, by introducing a dimensionless parameter, $\lambda_{c\phi}$, which is defined by

$$\lambda_{c\phi} = \frac{\gamma H \tan \phi}{c} \qquad (3)$$

The studies performed by Janbu showed that the results of slope stability analyses may be expressed uniquely in terms of $\lambda_{c\phi}$ and two other dimensionless coefficients, $F\gamma H/c$, and the tangent of the slope angle. Thus for any combination of $\lambda_{c\phi}$ and slope angle, the value of $F\gamma H/c$ calculated by any method is unique, i.e., any combination of γ, H, c, and ϕ which gives the same value of $\lambda_{c\phi}$ will result in the same value of $F\gamma H/c$. In addition, the distribution of normal stress on the shear surface as determined by any method is precisely similar for any two slopes having the same inclinations and values of $\lambda_{c\phi}$. Therefore, in comparing the results obtained by various methods, the effects of changes in γ, H, c, and $\tan\phi$ may all be combined in changes in the value of $\lambda_{c\phi}$.

The value of $\lambda_{c\phi}$ increases as the frictional component of the soil strength increases in comparison with the cohesive component. Possible values of $\lambda_{c\phi}$ range from zero for a slope with $\tan\phi = 0$, or completely cohesive strength, up to infinity for a slope with $c = 0$, or completely frictional strength. In this study, values of $\lambda_{c\phi}$ ranging from 0 to 50 were investigated, and slope angles ranging from 1.5 (horizontal) on 1 (vertical) to 3.5 on 1. These values encompass most of the conditions encountered in practical slope stability problems.

Normal Stress Distribution

The distributions of normal stresses determined by Bishop's Simplified Method for slopes of 1.5 on 1 and 3.5 on 1 and values of $\lambda_{c\phi}$ equal to 0 and 20 are shown in Fig. 1 by dashed lines. The distributions shown in the same figure by solid lines were determined from the linear elastic stress distributions for these same slopes, calculated by the finite element method. The finite element analyses were performed simulating construction of the slope in a series of layers, using the procedures described by Clough and Woodward (2). The value of Young's Modulus was assumed constant throughout the slope, and the value of Poisson's ratio was assumed to be 0.42.

It may be noted that the greatest difference between the normal stresses calculated by Bishop's Simplified Method and those determined from the linear elastic stress distribution corresponds to the steeper slope, and smaller value of $\lambda_{c\phi}$. For the flatter slope and larger value of $\lambda_{c\phi}$, the difference between the normal stresses calculated by the two methods is very small. For the cases where the differences are greatest, it may be seen that the normal stresses calculated by Bishop's Simplified Method are greater than those determined by the linear elastic stress distribution in the central portion of the arc, and smaller near the ends.

Variations of F Along Shear Surface

The factor of safety calculated by Bishop's Simplified Method is assumed to be the same for every slice, and is thus constant for every point on the shear surface. The values calculated from the linear elastic stress distribution, however, are not constant, as shown in Fig. 2. The curves in this figure show variations of the values of F calculated from linear elastic stress distributions for the same cases shown in Fig. 1. For the cases studied it was found that along about one-third to one-half of the shear surface the factors of safety calculated using the linear elastic stresses were less than the average value for the slope.

From these results it is possible to determine the minimum value of overall factor of safety required so that the factor of safety will not be less than unity at any point on the shear surface, i.e., the value of F required to prevent

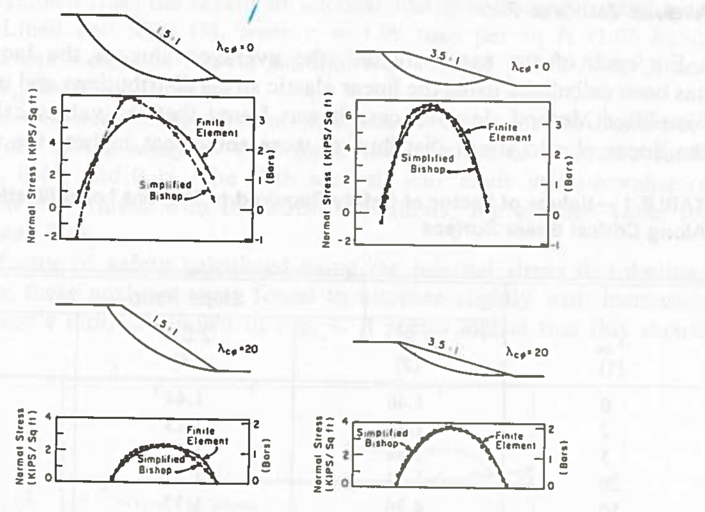

FIG. 1.—Variations in Normal Stressas along Shear Surfaca for Simplified Bishop and Finita Element Analyses

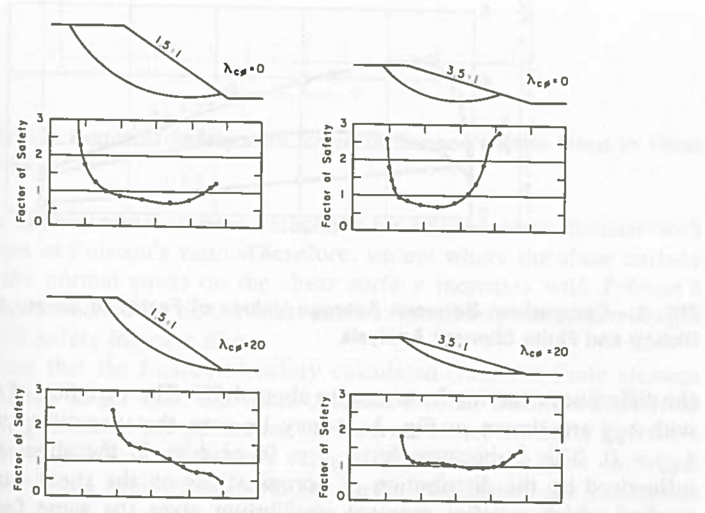

FIG. 2.—Variation in Factor of Safety along Shaar Surface from Finite Element Analyses (Simplified Bishop F = 1.0)

overstress according to linear elastic theory. These values are shown in Table 1. It may be seen that the value of F required to prevent the shear stresses on the shear surface from exceeding the shear strength of the soil vary from a maximum of 4.36 for a 1.5 on 1 slope and $\lambda_{c\phi}$ = 50, to a minimum of 1.10 for a 3.5 on 1 slope and $\lambda_{c\phi}$ = 20. For a wide range of conditions, a factor of safety equal to 1.5 is sufficiently large to prevent local elastic overstress.

Average Values of F

For each of the cases studied, the average value of the factor of safety has been calculated using the linear elastic stress distributions and using Bishop's Simplified Method. In each case it was found that the values calculated from the linear elastic stress distribution were somewhat higher, the magnitude of

TABLE 1.—Values of Factor of Safety Required to Prevent Local Elastic Overstress Along Critical Shear Surface

$\lambda_{c\phi}$ (1)	Slope Ratio 1.5:1 (2)	2.5:1 (3)	3.5:1 (4)
0	1.46	1.44	1.49
2	1.32	1.23	1.23
5	1.34	1.18	1.14
20	2.27	1.21	1.10
50	4.36	1.37	1.12

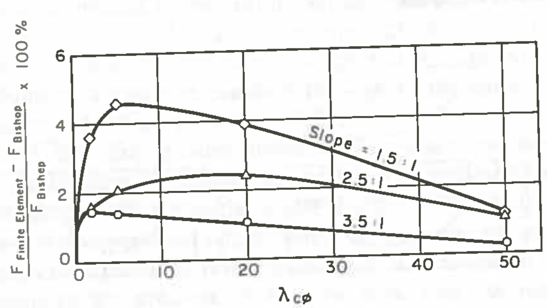

FIG. 3.—Comparison Between Average Values of Factor of Safety from Simplified Bishop and Finite Element Analysis

the differences varying from zero to about 4.5%. The variation of the differences with $\lambda_{c\phi}$ are shown in Fig. 3. It may be seen that the difference is zero for $\lambda_{c\phi}$ = 0. This is because for $\lambda_{c\phi}$ = 0, or ϕ = 0, the shear strength is not influenced by the distribution of normal stress on the shear surface, and any method which satisfies moment equilibrium gives the same factor of safety, regardless of the normal stress distribution. It may also be seen that as $\lambda_{c\phi}$ becomes large, the difference decreases. This is because as the value of $\lambda_{c\phi}$ increases, the critical shear surface becomes flatter, and the normal stresses calculated by the two methods become more nearly equal. As a result, the maximum difference in overall factor of safety corresponds to intermediate values of $\lambda_{c\phi}$.

Differences of about the same magnitude have also been found for a case in which the internal stresses were determined by nonlinear finite element analyses. These analyses were made for Otter Brook Dam (5), which is about 130 ft high with side slopes 2.5 on 1. The strength characteristics of the fill material, determined from the results of unconsolidated undrained triaxial tests reported by Linell and Shea (5), were c = 1.08 tons per sq ft (1.03 bars), ϕ = 14°. The unit weight of the fill material was 140 pcf (2,240 N/m^3), and the value of $\lambda_{c\phi}$ for slope is about 2.

In all, five nonlinear finite element analyses of Otter Brook Dam were performed. Four of the analyses were made using values of Poisson's ratio, ν = 0.3, 0.4, 0.44, and 0.49. The fifth analysis was made using a value of Poisson's ratio that varied with the stress conditions; the average value for this analysis was 0.45.

Values of factor of safety calculated using the internal stress distributions determined by these analyses were found to increase slightly with increasing value of Poisson's ratio, as shown in Fig. 4. It seems logical that this should

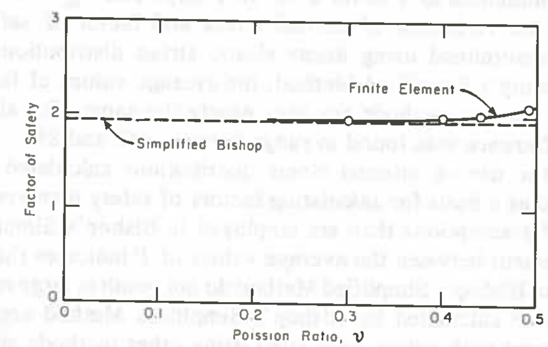

FIG. 4.—Variation in Factor of Safety with Value of Poisson's Ratio Used in Finite Element Analyses

occur, because the horizontal stresses calculated for embankments increase with increasing values of Poisson's ratio. Therefore, except where the shear surface is horizontal, the normal stress on the shear surface increases with Poisson's ratio, and as the normal stress on the shear surface increases, the shear strength and the factor of safety increase also.

It may be seen that the factors of safety calculated from the finite element stresses are slightly larger than the value calculated using Bishop's Simplified Method. The magnitude of the difference varies from about 2% for a Poisson's ratio of 0.3 up to 8% for a Poisson's ratio of 0.49. For the case in which the value of Poisson's ratio varies with the stress conditions, the difference is about 4%.

Conclusions

In the preceding sections, the normal stress distributions and the factors of safety determined using Bishop's Simplified Method have been compared with values determined using the internal stress distributions calculated using linear and nonlinear finite elements analyses. From the results of this study, the following conclusions may be drawn:

1. The normal stress distributions determined by linear elastic finite element analyses are very nearly the same as those determined by Bishop's Simplified Method for flat slopes and large values of $\lambda_{c\phi}$. The difference between the normal stress distributions determined by the two methods increases with increasing slope angle and decreasing values of $\lambda_{c\phi}$. For steep slopes and small values of $\lambda_{c\phi}$, the normal stresses determined by linear elastic stress analyses are smaller at the center of the critical circle, and larger at the end than the values determined by Bishop's Simplified Method.

2. Although the factor of safety is assumed to be the same for each slice in Bishop's Modified Method, values calculated using linear elastic stress distributions are not constant. For the cases studied, the value of F required to prevent the linear elastic stress from exceeding the shear strength of the soil at any point varies from a maximum of 4.36 for a 1.5 on 1 slope and $\lambda_{c\phi} = 50$, to a minimum of 1.10 for a 3.5 on 1 slope and $\lambda_{c\phi} = 20$.

3. Although the variations of normal stress and factor of safety along the shear surface determined using linear elastic stress distributions are not the same as for Bishop's Simplified Method, the average values of factor of safety determined by the two methods are very nearly the same. For all of the cases studied, the difference was found to range between 0% and 8%.

4. Because the use of internal stress distributions calculated by the finite element method as a basis for calculating factors of safety involves a completely different set of assumptions than are employed in Bishop's Simplified Method, the close agreement between the average values of F indicates that the assumptions involved in Bishop's Simplified Method do not result in large errors. Further, since values of F calculated by Bishop's Simplified Method are known to be in good agreement with values calculated using other methods such as Janbu's Generalized Procedure of Slices and Morgenstern and Price's Method, it may be concluded that none of these procedures involves large errors.

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