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ENGLISH-SI CONVERSION FACTORS

In accordance with the October 1970 action of the ASCE Board of Direction, which stated that all publications of the Society should list all measurements in both customary (English) and SI (International System) units, the list below contains conversion factors to enable readers to compute the SI unit values of measurements. A complete guide to the SI system and its use has been published by the American Society for Testing & Materials. Copies of this publication (ASTM E-380-1972) can be purchased from ASCE at a price of 75¢ each; orders must be prepaid.

All authors of Journal papers are being asked to prepare their papers in this dual-unit format. Until this practice affects the majority of papers published, we will continue to print this table of conversion factors:

To convert	To	Multiply by
inches (in.)	millimeters (mm)	25.40
inches (in.)	centimeters (cm)	2.540
inches (in.)	meters (m)	0.0254
feet (ft)	meters (m)	0.305
miles (miles)	kilometers (km)	1.61
yards (yd)	meters (m)	0.91
square inches (sq in.)	square centimeters (cm^2)	6.45
square feet (sq ft)	square meters (m^2)	0.093
square yards (sq yd)	square meters (m^2)	0.836
acres (acre)	square meters (m^2)	4047.
square miles (sq miles)	square kilometers (km^2)	2.59
cubic inches (cu in.)	cubic centimeters (cm^3)	16.4
cubic feet (cu ft)	cubic meters (m^3)	0.028
cubic yards (cu yd)	cubic meters (m^3)	0.765
pounds (lb)	kilograms (kg)	0.453
tons (ton)	kilograms (kg)	907.2
one pound force (lbf)	newtons (N)	4.45
one kilogram force (kgf)	newtons (N)	9.81
pounds per square foot (psf)	newtons per square meter (N/m^2)	47.9
pounds per square inch (psi)	kilonewtons per square meter (kN/m^2)	6.9
gallons (gal)	cubic meters (m^3)	0.0038
gallons (gal)	liter (dm^3)	3.8
acre-feet (acre-ft)	cubic meters (m^3)	1233.
gallons per minute (gpm)	cubic meters/minute (m^3/min)	0.0038
newtons per square meter (N/m^2)	pascals (Pa)	1.00

JOURNAL OF THE SOIL MECHANICS AND FOUNDATIONS DIVISION

Three-Dimensional Finite Element Analyses of Dams

By Guy Lefebvre,[1] James M. Duncan,[2] M. ASCE, and Edward L. Wilson,[3] M. ASCE

Introduction

In the few years since its development, the finite element method has been widely used for studying many different type soil mechanics problems. Up to the present time, virtually all of the analyses performed have been two-dimensional or axisymmetric, for two reasons: (1) Many soil mechanics problems can be closely represented by conditions of plane strain or axially symmetric deformation; and (2) the available three-dimensional finite element computer programs have required large amounts of computer time, even for meshes with few nodal points and elements, and as a result it has not been feasible to solve many three-dimensional problems.

Wilson (2) has recently developed a three-dimensional finite element computer program which is efficient enough so that it can be used to solve many three-dimensional problems at costs for computer time comparable to those for two-dimensional analyses. The availability of this program and the prospect of even more efficient programs in the near future will undoubtedly make it possible to solve three-dimensional problems routinely.

Even when three dimensional finite element analyses can be done on a regular basis, however, it will still be desirable to perform two-dimensional analyses for those conditions for which they are sufficiently accurate, because two-dimensional analyses will always require less man-time and less computer time than

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three-dimensional analyses. Thus it is of considerable interest to determine those conditions for which two-dimensional analyses can be effectively used. For example, in the case of embankment dams, two-dimensional (plane strain) analyses provide an accurate representation of conditions in centrally located transverse sections of long dams of uniform cross section, and there is no reason to use three-dimensional analyses to study the stresses or movements under such conditions. However, plane strain analyses may not provide a suitable representation of the transverse section for dams in steep-walled valleys, because of the effects of cross-valley arching. Similarly, there is doubt concerning the accuracy of either plane stress or plane strain analyses of the longitudinal sections of dams, as the thickness of a dam perpendicular to the longitudinal section (i.e., measured from upstream to downstream) varies from the top of the dam to the bottom.

The purpose of this paper is to show the accuracy with which two-dimensional analyses may be used to study stresses and movements in dams in V-shaped valleys. This is done by comparing the results of plane strain analyses of the transverse sections, and the results of both plane strain and plane stress analyses of the longitudinal sections, with the results of three-dimensional analyses of the dams. These comparisons are prefaced by a description of the characteristics of the elements used for the three-dimensional analyses, a summary of the assumptions employed in the analyses, and a summary of the results of the three-dimensional analyses.

Element Characteristics

A quadrilateral element of arbitrary geometry was used for the two-dimensional plane strain analyses. This element degenerates to a triangular shape in order to represent the geometry at the boundaries of the dam. The basic three-dimensional element used was an eight-node isoparametric element. The displacement field assumptions for these elements are given in Ref. 2. Basically the displacements are assumed to vary linearly. However, additional degrees-of-freedom have been used internally, which greatly improve the accuracy. Both of these types of elements have been incorporated into a general three-dimensional structural analyses program. Because of the general mesh generation options within the program, it was not necessary to develop a special program for dam analyses.

Three-Dimensional Analyses of Dams

Dams Analyzed.—Analyses were performed for dams in V-shaped valleys with three different valley-wall slopes, 1:1 (1 horizontal: 1 vertical), 3:1, and 6:1 as shown in Fig. 1. Although the valley wall slopes were different for each case, the fill slopes were the same (2.5:1) in all three cases. Each of the dams was 160 ft (49 m) high, and was represented by eight layers of elements of uniform thickness, as shown in Fig. 1.

Assumptions.—Linear elastic material properties were used in the analyses, with Young's modulus, E = 100 tons/ft^2 (9,580 kN/m^2), and Poisson's ratio, ν = 0.4. The unit weight of the fill, γ, was 125 pcf (19.6 kN/m^3). Clough and Woodward (1) have shown that values of stress, calculated using linear elastic stress-strain characteristics, vary in proportion to the unit weight of

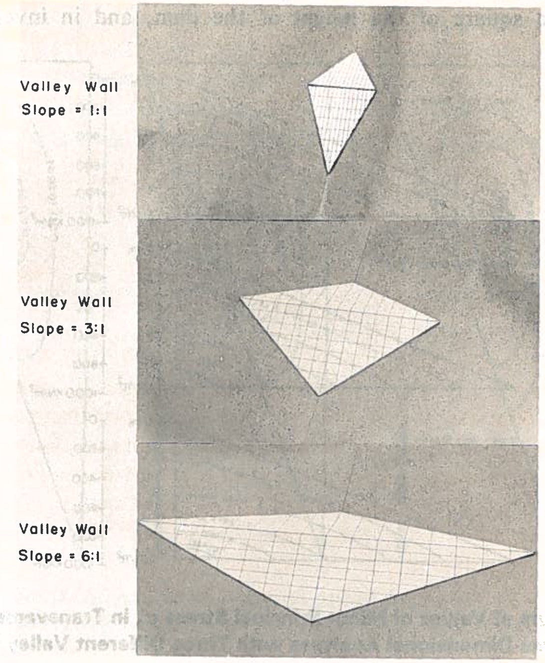

FIG. 1.—Dams Analyzed

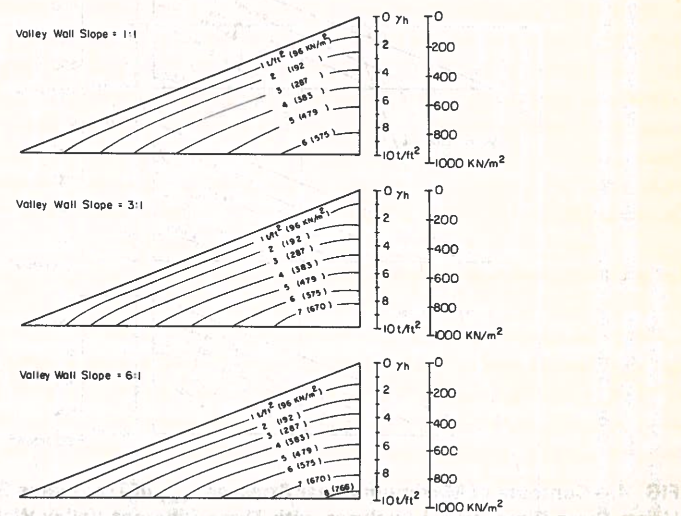

FIG. 2.—Contours of Values of Major Principal Stresses σ_1 in Transverse Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall slopes

the fill and the height of the dam, and are not affected by the value of Young's modulus. The values of calculated displacement vary in proportion to the unit weight and the square of the height of the dam, and in inverse proportion

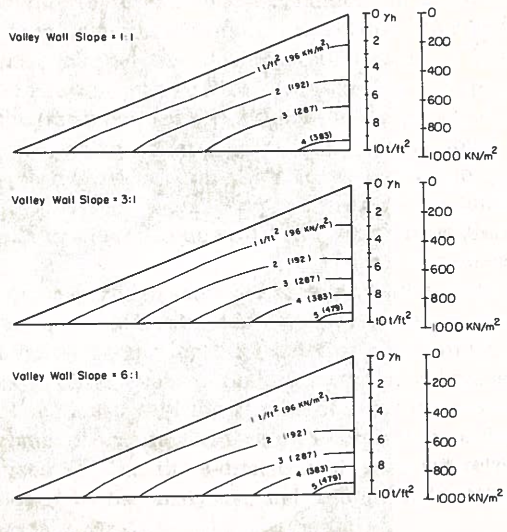

FIG. 3.—Contours of Values of Minor Principal Stress σ_3 in Transverse Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

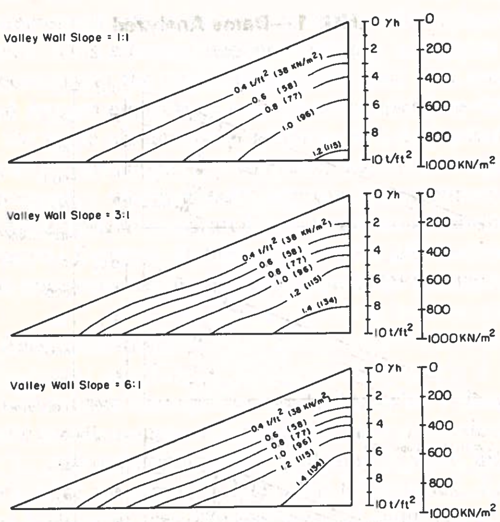

FIG. 4.—Contours of Maximum Shear Stresses τ_{max} in Transverse Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

to the value of Young's modulus. These facts may be used to derive from the results of a linear analysis the results for a dam of similar shape, but with

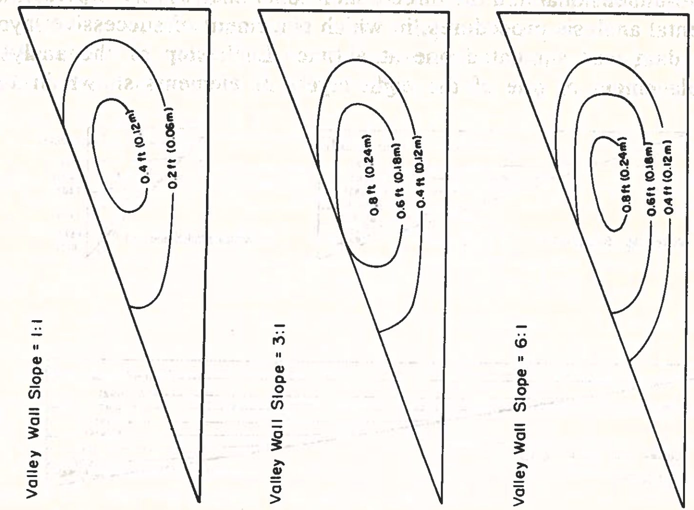

FIG. 6.—Contours of Values of Horizontal Displacement in Transverse Section u_h Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

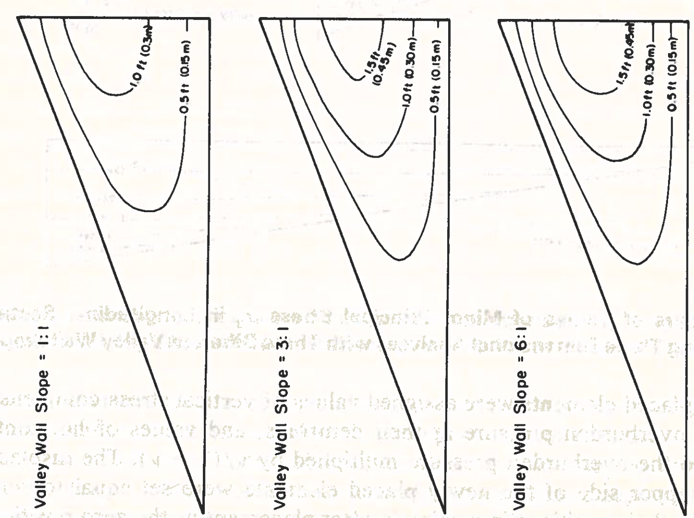

FIG. 5.—Contours of Values of Vertical Displacement in Transverse Section u_v Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

a different height and consisting of a material with different values of unit weight and Young's modulus. The value of Poisson's ratio used also affects the calculated stresses and displacements, but in a more complex manner.

Both the two-dimensional and the three-dimensional analyses were performed using incremental analysis procedures, in which placement of successive layers of fill in the dam was simulated one at a time. Each step of the analyses represented placement of one of the eight layers of elements shown in Fig.

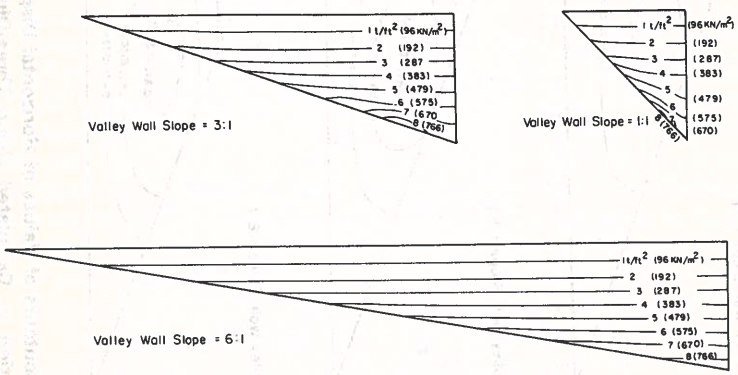

FIG. 7.—Contours of Values of Major Principal Stress σ_1 in Longitudinal Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

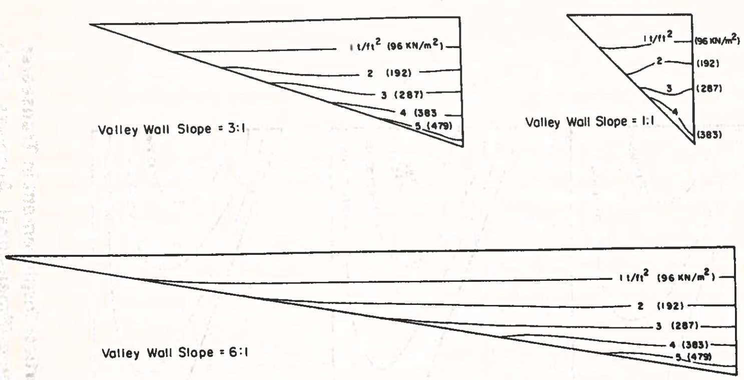

FIG. 8.—Contours of Values of Minor Principal Stress σ_3 in Longitudinal Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

1. The newly placed elements were assigned values of vertical stress commensurate with the overburden pressure at their centroids, and values of horizontal stress equal to the overburden pressure multiplied by $\nu/(1 - \nu)$. The displacements at the upper side of the newly placed elements were set equal to zero, in effect taking their positions immediately after placement as the zero positions for determination of movements. The nodal points along the bottom of the dam were fixed, simulating the condition of no slip between the dam and the adjacent valley wall.

Results for Transverse Sections.—Values of the stresses and displacements in the transverse section calculated in the three-dimensional analyses are shown in Figs. 2 through 6. In each figure, the results are shown for the three valley wall slopes analyzed, with the results for the 1:1 valley wall slope at the top, the 3:1 slope in the center, and the 6:1 slope at the bottom.

The effect of the valley wall slope on the values of major principal stress σ_1 may be seen in Fig. 2. (The values of σ_1 and σ_3 shown in Figs. 2 and

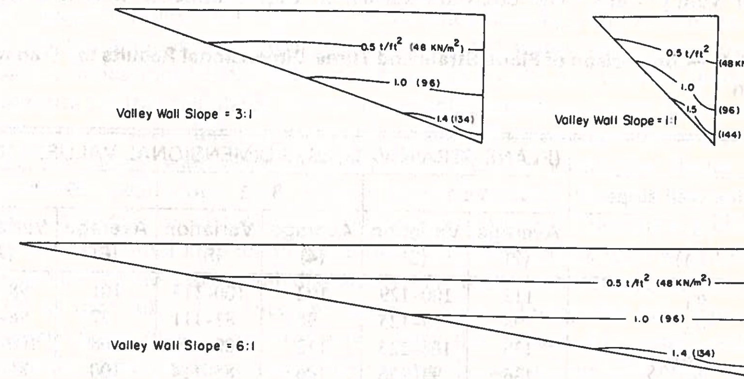

FIG. 9.—Contours of Values of Maximum Shear Stress τ_{max} in Longitudinal Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

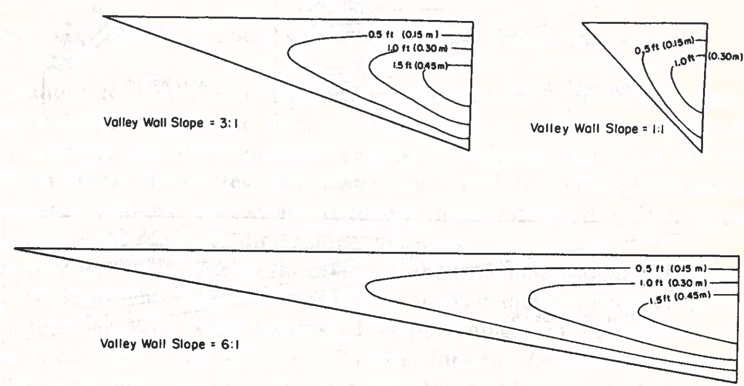

FIG. 10.—Contours of Values of Vertical Displacement u_v in Longitudinal Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

3 are the values of maximum and minimum principal stress in the transverse plane.) Although the values of σ_1 in the upper part of the dam are very nearly the same in all three cases, the values near the base of the dam are smaller for the steeper valley wall slopes, indicating a significant degree of cross-valley arching within the dams in the steeper valleys. The values of minor principal stress, σ_3, shown in Fig. 3 indicate a similar influence of valley wall slope, with the values near the base of the dam decreasing with increasing steepness of the valley wall. Similarly, the values of maximum shear stress, τ_{max}, shown

in Fig. 4 may be seen to follow the same pattern.

The settlements or vertical displacements, u_v, are shown by the contours in Fig. 5. These movements are due to the compression of the already placed material under the load of the material above, and thus correspond to the settlements or vertical movements in dams during construction. It may be seen that the magnitudes of the settlements decrease with increasing steepness of the valley wall, indicating a stronger tendency for the dam to "hang up" on the steeper valley walls. The contours shown in Fig. 6 indicate that the steeper

TABLE 1.—Comparison of Plane Strain and Three-Dimensional Results for Transverse Section

	(PLANE STRAIN VALUES/3-DIMENSIONAL VALUES) 100%					
Valley wall slope	1 : 1		3 : 1		6 : 1	
(1)	Average (2)	Variation (3)	Average (4)	Variation (5)	Average (6)	Variation (7)
σ_1	113	100-129	102	100-113	101	98-109
σ_3	98	79-125	96	81-111	97	88-100
τ_{max}	138	108-225	112	100-150	108	100-150
u_v	136	91-156	106	85-114	100	85-105
u_h	268	75-435	120	80-149	105	85-120

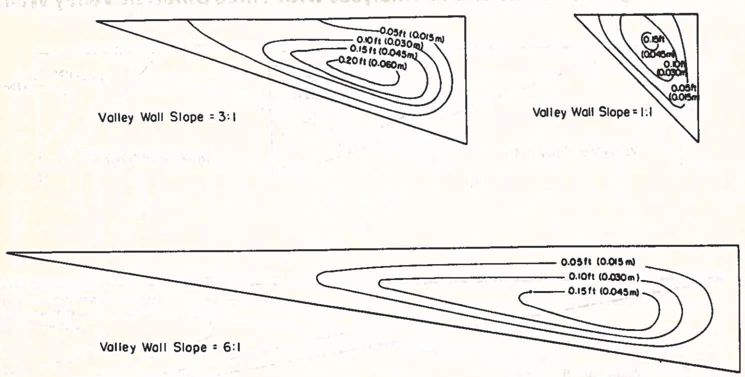

FIG. 11.—Contours of Values of Horizontal Displacement u_h in Longitudinal Section Calculated Using Three-Dimensional Analyses with Three Different Valley Wall Slopes

valley walls also tend to restrict the horizontal movements, u_h. These effects are consistent with the cross-valley arching noted previously in regard to the stresses in the dams.

Results for Longitudinal Sections.—Contours of the stresses and displacements in the longitudinal sections calculated in the three-dimensional analyses are shown in Figs. 7 through 11. It may be seen that the calculated values of σ_1, shown in Fig. 7, increase uniformly with depth except near the valley walls. (The values of σ_1 and σ_3 shown in Figs. 7 and 8 are the values of maximum and minimum principal stress in the longitudinal plane.) The values of σ_1 at

the bottom of the dam are smallest for the dam in the steepest valley, indicating some degree of cross-valley arching. The calculated values of σ_3, shown in Fig. 8, are affected in a similar way by valley wall slope. The calculated values of τ_{max}, shown in Fig. 9, reach slightly larger values for the dam in the steepest valley than for the dams in the flatter valleys.

Contours of settlements or vertical displacement u_v during construction are shown in Fig. 10. In each of the three cases shown the variations are similar, but the maximum value, near the center of the section, is smaller for the dam in the steepest valley as a result of cross-valley arching. The maximum values of u_v for the dams in the valleys with 3:1 and 6:1 valley wall slopes are about the same, indicating that the effects of the valley walls are small in these cases.

Contours of the calculated horizontal displacements, u_h, are shown in Fig. 11. It may be seen that the maximum value of u_h in the dam in the valley with a 3:1 valley wall slope is larger than that for either the flatter or the steeper valley wall slope. It is believed that this occurs because there are two counteracting effects of increasing valley wall slope: (1) There is a greater tendency for horizontal movement (note that if the base of the dam was horizontal there would be no horizontal displacements in the dam during construction); and (2) as the valley walls become steeper, they offer greater restraint against movement (note that if the dam was a very thin wedge in a valley with very steep walls approaching vertical, there would be almost no horizontal displacements in the longitudinal section during construction). As a result of these two counteracting influences, the magnitude of the horizontal displacement first increases and then decreases as the valley wall slopes become steeper.

Comparison of Results of Two-Dimensional and Three-Dimensional Analyses

In addition to the three-dimensional analyses described previously, two-dimensional analyses of the same dams were also performed. A plane strain analysis was performed for the transverse section, which was the same for all three cases, and both plane stress and plane strain analyses were performed for the longitudinal sections. The same procedures and material property values were employed as for the three-dimensional analyses.

For purposes of comparing the results of the two-dimensional and three-dimensional analyses, the values of stresses and displacements calculated by the two-dimensional analyses were expressed as percentages of the values calculated by the three-dimensional analyses. This was done for the stresses in each element and the displacements at each nodal point. The results were found to be consistent except for the nodal points on the slopes and the elements adjacent to the slopes, where the variations were extreme and erratic. Detailed study showed that these large variations occurred for two reasons: (1) The calculated values of stress near the slopes were small, and differences in values which were not large, compared with the significant stresses in the dams, nevertheless indicated large percentage differences; and (2) the triangular-shaped elements adjacent to the slopes were rather stiffer than appropriate, and therefore tended to restrict the deformations. It was considered that the best interpretation of the results could be made ignoring the values for these triangular elements and the nodal points along the slopes. Therefore, in the tables of values presented in subsequent sections, the results shown do not include the nodal points on the slopes or the stresses for triangular elements adjacent to the slopes.

Transverse Section—Comparison with Plane Strain.—The results for the three-dimensional analyses and plane strain analyses of the transverse section are compared in Table 1. The values shown are the average and the ranges of the plane strain values expressed as percentages of the results of the three-dimensional analyses. The extreme values in most cases correspond to locations within the section where the absolute values were relatively small, and are therefore not significant for most purposes. The average values shown are representative of conditions within a large part of the cross section and are considered to provide a simple and reasonable basis for comparison.

It may be noted that, on the average, the values of σ_1 calculated by the plane strain analyses are larger than those calculated by the three-dimensional analyses, and the values of σ_3 from the plane strain analyses are slightly smaller than those from the three-dimensional analyses. The values of $\tau_{max} = (1/2)(\sigma_1 - \sigma_3)$ calculated by the plane strain analyses are larger in all cases than the values calculated by the three-dimensional analyses. It may also be seen that the values of horizontal and vertical displacements calculated by plane strain analyses are larger on the average than those calculated by the three-dimensional analyses.

For the dam in the valley with a 6:1 valley wall slope, the percentage differences are quite small, and it may be concluded that the assumption of plane strain for such cases would introduce very little error in the calculated stresses and displacements. For the dam in the valley with 3:1 valley wall slopes, the percentage differences are larger, but are still small compared to inaccuracies in representing the actual stress-strain characteristics of the fill material, and so might be acceptable for most purposes. For the dam in the valley with 1:1 valley wall slopes, the inaccuracies resulting from the assumption of plane strain are quite large; the average values of τ_{max}, u_v, and u_h calculated by the plane strain analyses exceed those calculated by the three-dimensional analyses by 36% or more. Thus it may be concluded that plane strain analyses provide an acceptable degree of accuracy for most purposes for analyses of the transverse sections of dams in valleys with valley wall slopes of 3:1 or flatter, but the errors arising from assumed plane strain conditions for dams in valleys with steeper valley walls can be significantly larger.

Longitudinal Section—Comparison with Plane Stress.—The results for the three-dimensional analysis and plane stress analysis of the longitudinal section are compared in Table 2. It may be seen that the plane stress results are not in very good agreement with the three-dimensional results, and that the differences between the two analyses are essentially the same for all three valley wall slopes. The values of σ_1 calculated by the plane stress analyses exceed those from the three-dimensional analyses by about 10% in all cases. This difference may be attributed to the fact that, in making the plane stress analysis, it is assumed that the thickness of the dam, measured perpendicular to the section shown in Fig. 7, is constant, whereas in reality the thickness is smaller near the top. Consequently, the weight of the overlying material above any horizon in the dam is somewhat greater for the plane stress analyses than for the three-dimensional analyses, and as a result the values of σ_1 are also greater.

The values of minor principal stress calculated by the plane stress analyses are smaller than those from the three-dimensional analyses by 15% to 23%.

This difference is due primarily to the fact that the plane stress analysis is based on the assumption that there is no restraint to deformation in the direction normal to the section shown in Fig. 8. Because the values of σ_1 calculated by the plane stress analysis are larger and the values of σ_3 are smaller, the values of $\tau_{max} = (1/2)(\sigma_1 - \sigma_3)$ are considerably larger. It may be seen that the values from the plane stress analyses exceed those from the three-dimensional analyses by about 50% on the average for all three valley wall slopes.

It may also be noted that the values of horizontal and vertical displacement calculated by the plane stress analysis are greated than those calculated by

TABLE 2.—Comparison of Plane Stress and Three-Dimensional Results for Longitudinal Section

Valley wall slope	(PLANE STRESS VALUES/3-DIMENSIONAL VALUES) 100%					
	1 : 1		3 : 1		6 : 1	
(1)	Average (2)	Variation (3)	Average (4)	Variation (5)	Average (6)	Variation (7)
σ_1	109	80-127	110	102-115	111	100-115
σ_3	77	20-109	84	60-109	85	63-100
τ_{max}	149	107-185	149	111-179	149	126-181
u_v	160	122-196	173	138-217	173	140-224
u_h	220	130-1300	228	139-400	224	139-400

TABLE 3.—Comparison of Plane Strain and Three-Dimensional Analysis for Longitudinal Section

Valley wall slope	(PLANE STRAIN VALUES/3-DIMENSIONAL VALUES) 100%					
	1 : 1		3 : 1		6 : 1	
(1)	Average (2)	Variation (3)	Average (4)	Variation (5)	Average (6)	Variation (7)
σ_1	111	91-115	110	98-115	110	93-115
σ_3	122	85-130	124	100-136	123	100-135
τ_{max}	94	70-117	91	74-105	90	83-107
u_v	97	78-113	98	75-120	97	80-124
u_h	117	72-166	118	75-233	115	71-200

the three-dimensional analyses, the differences amounting to 60% to 70% for the horizontal displacements and 120% and 130% for the vertical displacements. These differences arise from the lack of restraint normal to the section for the plane stress analyses.

The rather large differences between values of stress and displacement calculated by means of plane stress and three-dimensional analyses indicates that plane stress analyses do not provide an accurate representation of the conditions in the longitudinal section of earth dams.

Longitudinal Section—Comparison with Plane Strain.—The results for the three-dimensional analyses and plane strain analyses of the longitudinal section are compared in Table 3. As was the case for the plane stress analyses, the values of σ_1 calculated assuming plane strain conditions exceed those from the three-dimensional analyses by about 10%. The reason is that, in the plane strain analysis, as in the plane stress analysis, the thickness of the section is assumed constant throughout its height and as a result the overburden at any level is greater for the plane strain analysis than for the three-dimensional analysis.

The values of minor principal stress, σ_3, calculated by the plane strain analyses are greater than those calculated by the three-dimensional analysis, by an average 22% to 24% for the three cases analyzed. These differences are believed to result from the fact that both the overburden and the restraint are larger in the plane strain analyses. The amount by which the value of σ_3 calculated by plane strain analyses exceed those calculated by three-dimensional analyses would be expected in increase as the value of Poisson's ratio for the dam material increased.

The values of maximum shear stress, τ_{max}, calculated by the plane strain analysis may be seen to be slightly smaller than the values for the longitudinal section calculated by the three-dimensional analyses. The difference is smallest for the steepest valley wall slope (an average of 6%) and largest for the flattest valley wall slope (an average of 10%).

The values of vertical displacement calculated by the plane strain analyses are 2% to 3% smaller than those calculated by the three-dimensional analyses, although the horizontal displacements from the plane strain analyses are 15% to 18% larger than those from the three-dimensional analyses.

Thus, although there are some differences in the values of stress and displacement in the longitudinal direction calculated by plane strain and three-dimensional analyses, it may be seen that the results are in rather good agreement for all three valley wall slopes. Therefore, even though the conditions in the longitudinal section might not be expected to correspond very closely to conditions of plane strain, the values of stress and displacement calculated by plane strain analyses are not much different from those calculated by the three-dimensional analyses.

Summary and Conclusions

Three-dimensional finite element analyses were performed for three dams in V-shaped valleys with three different valley wall slopes equal to 1:1, 3:1, and 6:1. These analyses were performed in increments, simulating construction of the dams in eight steps, using linear elastic material properties. The results of these analyses were compared with the results of incremental plane strain analyses of the maximum transverse section and incremental plane stress and plane strain analyses of the maximum longitudinal section. The results of the comparisons, summarized in Tables 1, 2, and 3, provide a basis for assessing the accuracy of the two-dimensional analyses for dams in V-shaped valleys. Consideration of these results has led to these conclusions:

1. For dams in valleys with valley walls inclined at 3:1 or flatter, plane strain analyses of the maximum transverse section will provide reasonably accurate results, but for dams in valleys with valley wall slopes as steep as 1:1, the results will be significantly less accurate, as a result of cross-valley arching.

2. Plane stress analyses of the maximum longitudinal section do not provide accurate results. The differences between the values calculated by the plane stress and the three-dimensional analyses were quite large and were of about the same magnitude for all three valley wall slopes studied.

3. Plane strain analyses of the maximum longitudinal section proved fairly accurate for all of the valley wall slopes analyzed. The greatest differences found were for the minor principal stresses, σ_3; the values of σ_3 calculated by plane strain analyses exceeded those calculated by the three-dimensional analyses by an average of about 23% for the three cases studied.

The experience gained during the course of this study indicates that it would be feasible to perform three-dimensional analyses of dams for practical purposes, if the conditions were such that two-dimensional analyses would not provide sufficient accuracy. With the computer program used in this study (2) three-dimensional analyses of dams may be performed using a sufficient number of elements so that accuracy is not impaired, without using unreasonable amounts of computer time.

Acknowledgment

The writers greatefully acknowledge the assistance of Terry Howard, member of the Soil Mechanics research staff, who performed a number of the analyses.

Appendix.—References

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